This article presents a model that is a natural generalization of both the spatial effects linear model and the linear model with spatial disturbances. Maximum likelihood methods are presented that provide estimates of the parameters of the model together with the asymptotic variance-covariance matrix of the estimates. Numerical illustrations of these methods are provided.

Maximum Likelihood Methods for Linear Models

Spatial Effect and Spatial Disturbance Terms

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This article is concerned with the formulation of, and estimation of, a certain class of linear models for social phenomena distributed across geographical space. These models have their origin in the classical population regression functions of regression analysis, and have a certain data structure. Essentially, the data for these models are obtained for a set of areas that comprise some region, with each area being construed as a unit of analysis. Examples of such phenomena are provided by Mitchell's (1969) analysis of the Huk rebellion in the Philippines, Chirot and Ragin's (1975) analysis of the Romanian peasant rebellion of 1907, Salamon and Van Evera's (1973) analysis of black political participation in Mississippi, and Frisbie and Poston's (1975) analysis of sustenance organization. Further examples are discussed by Doreian (1981).

When regression analysis is employed, variables characterizing the areas are used to account for the variation of an endogenous variable across the areas comprising a region. Among the assumptions underlying the use of such a method is one to the effect that the observations are independent. However, with social phenomena distributed in geographical space, this assump-

SOCIOLOGICAL METHODS & RESEARCH, Vol. 10 No. 3, February 1982 243-269

tion may be questioned insofar as the value of the endogenous variable in a particular area is dependent upon the values of that variable in adjacent or nearby areas. This interdependency must be included in the specification of a model of the phenomenon being considered. Ord (1975) presented a general method of doing this through the specification of and use of a matrix W that represents the spatial structure of a region. Essentially, this matrix represents some relation of interest defined over the areas that make up the region. Within Ord's general approach, there are distinct models that can be considered. Doreian (1981) discusses the spatial effects model (and includes also a discussion of the specification of W). The spatial disturbances model (Doreian, 1980) is another model within Ord's approach. This article presents a model that is a natural extension of both models together with an estimation strategy for estimating that model.

A LINEAR MODEL WITH A SPATIAL EFFECT AND A SPATIAL DISTURBANCE TERM

The specification of a model where the value of the endogenous variable for one area of geographic space is interdependent with the values of that variable for some other areas making up the overall region is the spatial effects model:

$$Y = \rho_1 W_1 Y + X\beta + \epsilon \qquad [1]$$

where W_1 is an appropriate matrix of weights; ρ_1 is the spatial effects parameter; X is a matrix of observations for the exogenous variables (including a column of 1s for the intercept); β is a vector of parameters to be estimated, and ϵ is a disturbance term. If there are N areas making up the region of interest, then W_1 is an $(N \times N)$ matrix. Both Y and ϵ are $(N \times 1)$ vectors. If there are

(K − 1) exogenous variables, then X is an (N × K) matrix (including a column of 1s for the intercept term), and β is a (K × 1) vector of parameters. If $\epsilon \sim IN(O, \sigma^2 I)$, then we have the model studied extensively by Ord (1975) and Doreian (1981). However, ϵ may also be spatially autoregressive:

$$\epsilon = \rho_2 W_2 \epsilon + \nu \qquad [2]$$

with

$$E\nu\nu' = \sigma^2 I \qquad [3]$$

and ν an (N × 1) vector and multivariate normal. The matrix W_2 is another matrix of weights (although $W_2 = W_1$ is possible), and ρ_2 is a spatial parameter for the disturbance term.[1] If $\rho_1 = 0$, then we have the spatial disturbance model studied by Ord (1975) and by Doreian (1980).

This article establishes a maximum likelihood (ML) procedure for estimating the parameters of this model having both a spatial effects term and a spatial disturbances term together with the variance-covariance matrix for these estimators. The probabilistic information is contained in the assumption concerning the distribution of ν. However, ν is unobserved. In order to obtain the ML procedure, it is necessary to perform two transformations (from ν to ϵ and from ϵ to Y, given Y = y),[2] so as to have an expression for the likelihood function for the observed y. The likelihood function for the ν is given by:

$$L(\nu) = \left(\frac{1}{2\pi\omega}\right)^{N/2} \exp\left\{-\left(\frac{1}{2\omega}\right)\nu'\nu\right\}$$

where $\omega = \sigma^2$ to simplify notation. Two matrices are useful for expressing the probability distribution of the observed y rather than ν: $A_1 = I - \rho_1 W_1$ and $A_2 = I - \rho_2 W_2$. First $\nu = (I - \rho_2 W_2)\epsilon = A_2\epsilon$

and, from [1], $\epsilon = (I - \rho_1 W_1) y - X\beta = A_1 y - X\beta$. When the ν are transformed to the ϵ we have:

$$L(\epsilon) = |A_2| \left(\frac{1}{2\pi\omega}\right)^{N/2} \exp\left\{-\left(\frac{1}{2\omega}\right)(A_2\epsilon)'(A_2\epsilon)\right\}$$

$$= |A_2| \left(\frac{1}{2\pi\omega}\right)^{N/2} \exp\left\{-\left(\frac{1}{2\omega}\right)\epsilon' A_2' A_2 \epsilon\right\} \qquad [5]$$

where $|A_2|$ is the Jacobian of the transformation. Finally, from the transformation of the ϵ to y, given Y = y for a set of observations,

$$L(y) = |A_2|\,|A_1| \left(\frac{1}{2\pi\omega}\right)^{N/2}$$

$$\exp\left\{-\frac{1}{2\omega}(A_1 y - X\beta)' A_2' A_2 (A_1 Y - X\beta)\right\} \qquad [6]$$

where $|A_1|$ is the Jacobian of the transformation from the ϵ to the y. The log-likelihood function[3] is:

$$L(y) = \text{const} - \left(\frac{N}{2}\right)\ln\omega - \frac{1}{2\omega}[(A_1 y - X\beta)' A_2' A_2 (A_1 y - X\beta)]$$

$$+ \ln|A_1| + \ln|A_2| \qquad [7a]$$

Equivalently (on multiplying out the expression in square parentheses):

$$L(y) = \text{const} - \left(\frac{N}{2}\right)\ln\omega - \frac{1}{2\omega}[y' A_1' A_2' A_2 A_1 y$$

$$- 2\beta' X' A_2' A_2 A_1 y + \beta' X' A_2' A_2 X\beta] + \ln|A_1| + \ln|A_2| \qquad [7b]$$

At some points it will be convenient to use:

$$L(y) = \text{const} - \left(\frac{N}{2}\right) \ln\omega - \frac{1}{2\omega} [\epsilon' A_2' A_2 \epsilon]$$
$$+ \ln|A_1| + \ln|A_2| \qquad [7c]$$

in the derivations as $\epsilon' A_2' A_2 \epsilon$ is equal to the term in square parentheses in [7b] and in [7a]. We turn now to establishing the estimation equations.

From [7b]:

$$\frac{\partial L}{\partial \beta} = -\frac{1}{2\omega} [-2X' A_2' A_2 A_1 y + 2X' A_2' A_2 X\beta] \qquad [8]$$

Setting [8] to zero gives:

$$\hat{\beta} = (X' A_2' A_2 X)^{-1} X' A_2' A_2 A_1 y \qquad [9]$$

If ρ_1 and ρ_2 were known, then $\hat{\beta}$ can be regarded as the generalized least squares (GLS) estimator:

$$\hat{\beta} = (X' \Omega^{-1} X)^{-1} X' \Omega^{-1} y_1$$

where $\Omega^{-1} = A_2' A_2$ and $y_1 = A_1 y$. From [7c]:

$$\frac{\partial L}{\partial \omega} = -\frac{N}{2\omega} + \frac{1}{2\omega^2} [\epsilon' A_2' A_2 \epsilon] \qquad [10]$$

Setting [10] to zero gives:

$$\hat{\sigma}^2 = \hat{\omega} = \frac{1}{N} (\epsilon' A_2' A_2 \epsilon) \qquad [11]$$

In terms of the model parameters and observables, this is:

$$\hat{\omega} = \frac{1}{N} [y' A_1' A_2' A_2 A_1 y - 2\beta' X' A_2' A_2 A_1 y + \beta' X' A_2' A_2 X\beta] \qquad [12]$$

While [9] and [12] provide estimating equations for $\hat{\beta}$ and $\hat{\omega}$, they depend on ρ_1 and ρ_2 which are, in general, unknown.

There are two strategies for obtaining estimates of ρ_1 and ρ_2. The less computationally burdensome procedure is described here.[4] In order to minimize L(y) with respect to ρ_1 and ρ_2, the equations for $\partial L/\partial\rho_1 = 0$ and $\partial L/\partial\rho_2 = 0$ need to be solved. The left-hand side of these equations is given by [A.2] and [A.5] (see Appendix). Note that these equations cannot be solved directly as they contain ϵ and ω. In an empirical situation, $\hat{\epsilon} = A_1 y - X\hat{\beta}$ and, in principle, can be substituted for ϵ in [A.5] and used in [11] for $\hat{\omega}$. However, this expression contains A_1 and $\hat{\beta}$, neither of which are known until ρ_1 and ρ_2 have been estimated. Thus, expressions for these terms have to be substituted into [A.2] and [A.5] also. What results are two highly nonlinear equations that must be both set to zero and solved numerically.[5] One method of solving these equations numerically is the Newton-Raphson procedure (see Nielsen, 1968: 205-213 for a discussion of this procedure applied to systems of two nonlinear equations). With $\hat{\rho}_1$ and $\hat{\rho}_2$ found numerically, then $\hat{\beta}$ is found from [9] and $\hat{\omega}$ from [12].

The asymptotic variance-covariance matrix, V, for the parameter estimates is given by:

$$V^{-1} = -E\left[\frac{\partial^2 L}{\partial\theta_r \partial\theta_s}\right] \qquad [13]$$

where θ_r and θ_s denote pairs of parameters being estimated (Kendall and Stuart). The algebraic derivation of the right-hand side of [13] is provided in the Appendix. Using the notation

$$B_1 = A_2 W_1 A_1^{-1} A_2^{-1}$$

$$B_2 = W_2 A_2^{-1}$$

$$C = A_1'^{-1} W_1' A_2' A_2 W_1 A_1^{-1}$$

$$V_2 = W_2' + W_2 - 2\rho_2 W_2' W_2$$

$$D = A_2'^{-1} A_1'^{-1} W_1 V_2 A_2^{-1}$$

$$\alpha_1 = -\sum_{i=1}^{N} \lambda_i^2/(1 - \rho_1 \lambda_i)^2$$

and

$$\alpha_2 = -\Sigma \mu_i^2/(1 - \rho_2 \mu_i)^2$$

the variance-covariance matrix for this model is:

$$V(\hat{\omega}, \hat{\rho}_1, \hat{\rho}_2, \hat{\beta}) = \omega^2 \begin{bmatrix} N/2 & \omega \mathrm{tr} B_1 & \omega \mathrm{tr} B_2 & 0' \\ \omega \mathrm{tr} B_1 & \omega^2(\mathrm{tr} B_1' B_1 - \alpha_1) + \omega \beta' X' C X \beta & \omega^2 \mathrm{tr} D & (\omega X' A_2' A_2 W_1 A_1^{-1} X \beta)' \\ \omega \mathrm{tr} B_2 & \omega^2 \mathrm{tr} D & \omega^2(\mathrm{tr} B_2' B_2 - \alpha_2) & 0' \\ 0 & \omega X' A_2' A_2 W_1 A_1^{-1} X \beta & 0 & \omega X' A_2' A_2 X \end{bmatrix}^{-1} \quad [14]$$

As a check on this derivation, it should be noted that if $\rho_1 = 0$ and if the second row and the column are deleted (which correspond to ρ_1), then [14] reduces to the asymptotic variance-covariance matrix for the spatial disturbances model considered by Ord (1975) and Doreian (1980). Similarly, if $\rho_2 = 0$ and if the third row and the third column of [14] are deleted, then [14] reduces to the asymptotic variance-covariance matrix for the spatial effects models considered by Ord (1975) and by Doreian (1981). Similarly, the estimation equations reduce to their corresponding ML estimation equations when both $\rho_1 = 0$ and $\rho_2 = 0$.

In summary, the ML estimation procedure detailed herein is one that is performed in stages. Although the derivation of the ML procedure first established the estimation equations for $\hat{\beta}$ and $\hat{\omega}$ (analogously to OLS), and then moved to the estimation of $\hat{\rho}_1$ and $\hat{\rho}_2$, the ML procedure moves in reverse. First, the nonlinear equations obtained from setting [A.2] and [A.5] to zero are solved numerically.[6] Then, with ρ_1 and ρ_2 established, [9] and [11] are used to provide estimates of β and ω respectively. Finally, the variance-covariance matrix for the estimators is obtained from [14]. We turn now to some examples.

EXAMPLES

The numerical examples considered here are based on artificially constructed data.[7] In each case, data were generated according to the regime specified in [1], [2], and [3]. At issue is whether or not the estimation strategies detailed in the preceding section return the parameters used to generate the data (within tolerances due to sampling variability).

The first data sets conform to the example of Louisiana politics described in Doreian (1981). The dependent variable of interest is the proportion of voters in a parish (county) voting for the Democratic presidential candidate in 1960. Howard and Grenier (1978), by means of factorial ecology, delineated several dimensions that are important characteristics for describing the parishes of Louisiana. The data on three of these, percentage Black (X_1), percentage Catholic (X_2), and percentage urban (X_3) are, together with data for a measure of Black political equality (X_4), used as exogenous variables to generate an endogenous variable. The parameters used to generate the data are (respectively) 0, 0.25, 0.2, and 0.3 together with 15.0 for the intercept.[8]

The W matrices are constructed as follows. The matrix W_1 simply expresses contiguity of areas. Let $S_1 = [s_{ij}]$ be a matrix whose entries are 0 or 1. The entry $s_{ij} = 1$ if and only if area i is adjacent to area j and $s_{ii} = 0$ for all i. Then $w_{ij} = s_{ij}/s_{i.}$ where $s_{i.}$ is the i^{th} row sum of S_1. The rationale for this representation is discussed fully by Doreian (1981). The construction of W_2 is suggested by Howard's (1971) typology of parish types. The typology is based on a variety of geographic, geological, economic, social, and political criteria.[9] A matrix $S_2 = [s_{ij}]$ can be constructed where $s_{ij} = 1$ if and only if i and j belong to the same parish type. $W_2 = [w_{ij}]$ where $w_{ij} = s_{ij}/s_{i.}$ with $s_{i.}$ being the i^{th} row sum of S_2. The notion here is that parishes belonging to the same

TABLE 1
Parameter Estimates and Standard Errors using MLE Procedures for Various Generating Parameter Configurations

Parameters for Exogenous Variables	β Generating Values	(ρ_1, ρ_2) Generating Values			
		(.6, .5)	(.6, 0)	(0, .5)	(.3, .3)
ρ_1	--	0.64 (0.03)*	0.63 (0.03)	0.06 (0.07)	0.35 (0.05)
ρ_2	--	0.42 (0.13)	-0.06 (0.22	0.41 (0.14)	0.21 (0.17)
β_0	15.0	7.98 (3.72)	8.57 (3.07)	9.25 (3.52)	8.70 (3.34)
β_1	0	0.05 (0.04)	0.06 (0.04)	0.05 (0.04)	0.06 (0.04)
β_2	0.25	0.22 (0.03	0.22 (0.03)	0.23 (0.03)	0.23 (0.03)
β_3	0.20	0.22 (0.02)	0.23 (0.02)	0.22 (0.02)	0.22 (0.02)
β_4	0.03	0.32 (0.03)	0.32 (0.03)	0.33 (0.03)	0.33 (0.03)
ω	16.00	14.73 (2.65)	13.80 (2.45)	15.09 (2.71)	14.37 (2.53)

*Figures in parentheses are standard errors.

parish type behave similarly with respect to the endogenous variable of interest. In the following, several pairs of (ρ_1, ρ_2) were used to generate data: these were (.6, .5); (.6, 0); (0, .5); and (.3, .3). The estimation outcomes are shown in Table 1.

In terms of estimating ρ_1, the estimation procedure works well. In all cases, $\hat{\rho}_1$ is slightly above the generating value, but the largest discrepancy is 0.06 and all discrepancies are well within the estimated sampling variability. The estimated standard errors for ρ_1 are all small. The procedure appears to be less adequate when we turn to the estimates of ρ_2. In all cases, the returned estimate of ρ_2 was lower than the value used to generate the data with the maximum discrepancy being 0.09. Again, these lie within the estimated sampling variability. However, for ρ_2 the estimated standard errors are considerably higher than those that correspond to ρ_1. For the case where $\rho_1 = 0.3$ and $\rho_2 = 0.3$, the inferential decision concerning the presence of spatial autocorrelation in the disturbance term would be to omit it.[10] Even if an estimate of 0.3 was returned, this decision would stand. It may be that, for this kind of model, a value of $\hat{\rho}_2$ of around 0.3 may be the smallest that can be detected and dealt with. This suggests that values smaller than this may be ignored, as is the case for first order autoregressive processes (Hibbs, 1974). It should be noted that a generating value of $\rho_1 = 0$ was detected that would lead the researcher to a spatial disturbances model, and in the run for $\rho_2 = 0$, the absence of spatial autocorrelation in the disturbance term was detected that would lead the researcher to the spatial effects model.

We consider now the estimates of the parameters contained in β. The terms other than the intercept are considered first. The coefficient β_1 used in generating the data was zero and the estimates of this coefficient in the four runs are either 0.05 or 0.06. With an estimated standard error of 0.04 in each case, the null hypothesis that $\beta_1 = 0$ could not be rejected. The zero coefficient used in generating the data was detected. For all of the non-zero β_i used in generating the data, the returned estimates are close to their corresponding generating values. The estimated standard errors in each case are small (and identical to two places of decimals across all runs). Thus, as far as estimating the regression parameters, other than the intercept, the proposed ML procedure appears to work very well.

The results are less happy as far as the intercept is concerned. In all cases the returned estimates are low compared to the generating value. It is worth noting, however, that all estimates of the intercept are at least twice their corresponding standard errors: in none of these cases would the researcher be lead to infer a zero intercept. Nonetheless, at face value, there does appear to be a downward bias in the estimate of the intercept. If the "true" intercept is relatively large, there appears to be no danger that the researcher would erroneously conclude that it takes a zero value. If the precise value of the intercept is of interest, there is a limitation in the procedure being proposed. Monte Carlo studies will demonstrate whether or not there is truly a persistent downward bias in the estimate of the intercept, and if so, they will provide some guidance as to its magnitude.[11] If the "true" value of the intercept is small, there would appear to be some danger of its value being inferred as being zero.

The estimates of ω, the variance of the disturbance term, are also all quite reasonable. A crude measure of fit is 1 minus the ratio of the variance of the residual over the variance of the endogenous variable (Ord, 1975). Given the interdependencies among the variables, this is not a proportion of variance explained measure, although it is bounded by 0 and 1. It provides some guidance concerning the fit of the model. For the examples in Table 1 this measure is, reading from left to right, 0.99, 0.99, 0.96, and 0.98, respectively. Of course, given the nature of the constructed data these measures are not surprising. A full Monte Carlo study would indicate more clearly the adequacy of the procedure. However, it does appear that the estimation procedure proposed herein is a feasible one for dealing with both spatial effects and spatial disturbances in a linear model.

When the original body of data that suggested these simulated data was considered (Doreian, 1981), the estimation procedure for ρ_1 and ρ_2 returned $\hat{\rho}_1 = 0.31$ and $\hat{\rho}_2 = -0.14$. When the standard errors were estimated, the inferential decision concerning ρ_2 was that it was not significantly different from 0. This would lead the researcher back to the spatial effects model already estimated by Doreian (1981). By itself, this is encouraging. There is another view of the negative estimate of ρ_2. From the substantive context it is clear that both ρ_1 and ρ_2 should be bounded by the interval (0, 1). Both should be positive and bounded above by the reciprocal of the maximum eigenvalue of their corresponding W, as indicated earlier. A solution outside the permitted range may be interpreted as a diagnostic that the data do not conform to the kind of model specified here. In this case, ρ_2 is outside the permitted range which would again lead the researcher back to the spatial effects model.

The foregoing examples all involved data generated with distinct W_1 and W_2 matrices. There is no reason, in principle, why these must be distinct and the following examples uses $W_1 = W_2$.

TABLE 2
Parameter Estimates and Standard Errors for Data with $W_1 = W_2$

Parameter	Generating Value	Parameter Estimate	Standard Error
ρ_1	0.7	0.69	0.05
ρ_2	0.6	0.55	0.15
β_0	1.0	0.32	1.42
β_1	2.0	2.03	0.26
β_2	-1.0	-0.87	0.14
β_3	0.0	-0.03	0.06
β_4	30.0	29.51	2.00
β_5	12.0	15.20	2.28
ω	16.0	13.52	2.60

The data correspond to the Huk rebellion example (Mitchell, 1969; Doreian, 1981: Table 1) in which there are 5 actual exogenous variables used to generate the data. Both W_1 and W_2 are the (same) contiguity matrix constructed for the 57 regions of the Central Luzon studied by Mitchell. The parameter values for the exogenous variables used are 2, −1, 0, 30, and 12 together with an intercept of 1. The artificial endogenous variable is constructed in the same fashion as the foregoing examples (with $W_1 = W_2$) using the values $\rho_1 = 0.7$ and $\rho_2 = 0.6$. The resulting data were subjected to the proposed ML estimation procedure with the results shown in Table 2.

The same patterns are seen in this example as were seen in the foregoing examples. The estimate of ρ_1 is very good and the estimate of ρ_2 is slightly below the generating value with a discrepancy of 0.05 compared to the value used to generate the data. The estimate of the intercept again appears biased downward and is seen to be insignificantly different from zero. Note that a small value of β_0 was used in generating the data. The estimates of the regression coefficients are all close to their corresponding generating values being well within the bounds of sampling variability and the one coefficient that is 0 is detected as 0. The measure of fit described earlier takes a value of 0.97. Again, the procedure appears to work well even when the structure matrix for the spatial effects term is the same as the matrix for the spatial disturbance term.

CONCLUSION

This paper has suggested a linear model with both a spatial effects term and a spatial disturbances term as a natural generalization of the two linear models that have each of these terms separately. A maximum likelihood procedure was proposed for estimating this model and some examples of its use were provided using artificial data. Although not subjected to a full Monte Carlo simulation, the examples suggest that the procedure works as well for data generated by processes that conform to the specification of the model.

There do remain some problems. The results suggest that there is some value of ρ_2 (around 0.3) where a spatial disturbance cannot be detected. However, this may not be at all serious in practical situations where the estimation of β, ω, and the ρ_1 are of central importance. At the other extreme, a simulation was tried where ρ_1 and ρ_2 took values close to their maximum values; (the actual values used were $\rho_1 = \rho_2 = 0.9$). The ML estimation worked well in estimating ρ_1 and ρ_2 (where $\hat{\rho}_1 = 0.91$, $\hat{\rho}_2 = 0.89$), but the estimates of β and ω were very poor. This suggests there is a ceiling for the practical range[12] for which the ρs can be estimated and the entire estimation procedure implemented. One practical problem to be solved is to know better the practical bounds for the spatial parameters. Again, a Monte Carlo study will provide evidence on this issue.

With the knowledge that the model and estimation procedure work well for well-behaved data, several further steps are in order. One is the establishment of diagnostic procedures. A reasonable approach can be based on the work of Cliff and Ord (1973). As a first step, spatial autocorrelation can be tested for using W_1 (see Cliff and Ord, 1973: 13-15, 29-33). If spatial autocorrelation is present, then a spatial effects model can be fitted and the residual tested for spatial autocorrelation using W_2 (see Cliff and Ord, 1973: 87-97). If spatial autocorrelation is present in the residual, then the full spatial effects-spatial disturbances model can be fitted. Alternatively, inference concerning the ρ could be based on likelihood ratio tests (compare the approach of Brandsma and Ketellapper [1979] for two regimes of spatial autocorrelation in the disturbance term). A second avenue of inquiry is the determination of when this full approach is warranted and when a less complex procedure would be an adequate surrogate.

APPENDIX

DERIVATION OF VARIANCE-COVARIANCE MATRIX FOR THE ESTIMATORS

The log-likelihood function is given by [7a]-[7c]. From [7c] we have, writing L for L(y):

$$\frac{\partial L}{\partial \rho_1} = -\frac{1}{2\omega}\frac{\partial}{\partial \rho}(\epsilon' A_2' A_2 \epsilon) + \frac{\partial}{\partial \rho}\ln|A_1| \qquad [A.1]$$

$$\frac{\partial}{\partial \rho_1}(\epsilon' A_2' A_2 \epsilon) = \frac{\partial}{\partial \rho_1}(\epsilon') A_2' A_2 \epsilon + \epsilon' A_2' A_2 \frac{\partial}{\partial \rho_1}(\epsilon)$$

As

$$\epsilon = A_1 Y - X\beta = (I - \rho_1 W_1)y - X\beta, \quad \frac{\partial}{\partial \rho_1}(\epsilon) = -W_1 y$$

and

$$\frac{\partial}{\partial \rho_1} (\epsilon') = - y'W_1'$$

Thus

$$\frac{\partial}{\partial \rho_1} (\epsilon' A_2' A_2 \epsilon) = - y'W_1' A_2' A_2 (A_1 y - X\beta) - (y'A_1' - \beta' X') A_2' A_2 W_1 y$$

$$= \ldots = -2y'W_1' A_2' A_2 y + 2\rho_1 y'W_1' A_2' A_2 W_1 y + 2y'W_1' A_2' A_2 X\beta$$

From the expression for $|A_1|$ (footnote 3):

$$\frac{\partial}{\partial \rho_1} (\ln|A_1|) = - \sum_{i=1}^{N} \lambda_i/(1 - \rho_1 \lambda_i)$$

Substituting these into [A.1] gives:

$$\frac{\partial L}{\partial \rho_1} = \frac{1}{\omega} [y'W_1' A_2' A_2 y - \rho_1 y'W_1' A_2' A_2 W_1 y - y'W_1' A_2' A_2 X\beta]$$

$$- \sum_{i=1}^{N} \lambda_i/(1 - \rho_1 \lambda_i) \qquad \text{[A.2]}$$

From [A.2]:

$$\frac{\partial^2 L}{\partial \rho_1^2} = - \frac{1}{\omega} [y'W_1' A_2' A_2 W_1 y] - \Sigma \lambda_i^2/(1 - \rho_1 \lambda_i)^2 \qquad \text{[A.3]}$$

From [7c]:

$$\frac{\partial L}{\partial \rho_2} = - \frac{1}{2\omega} \frac{\partial}{\partial \rho_2} [\epsilon' A_2' A_2 \epsilon] + \frac{\partial}{\partial \rho_2} (\ln|A_2|) \qquad \text{[A.4]}$$

$$\frac{\partial}{\partial \rho_2}(\epsilon' A_2' A_2 \epsilon) = \epsilon' \frac{\partial}{\partial \rho_2}(A_2') A_2 \epsilon + \epsilon' A_2' \frac{\partial}{\partial \rho_2}(A_2)\epsilon$$

$$= -\epsilon' W_2'(I - \rho_2 W_2)\epsilon - \epsilon'(I - \rho_2 W_2') W_2 \epsilon$$

$$= -[\epsilon' W_2' \epsilon - \rho_2 \epsilon' W_2' W_2 \epsilon + \epsilon' W_2 \epsilon - \rho_2 \epsilon' W_2' W_2 \epsilon]$$

$$= -2[\epsilon' W_2 \epsilon - \rho_2 \epsilon' W_2' W_2 \epsilon], \text{ as } \epsilon' W_2' \epsilon = \epsilon' W_2 \epsilon$$

From the expression for $|A_2|$ (footnote 3):

$$\frac{\partial}{\partial \rho_2}(\ln|A_2|) = -\sum_{i=1}^{N} \mu_i/(1 - \rho_2 \mu_i)$$

Substituting these into [A.4] gives:

$$\frac{\partial L}{\partial \rho_2} = \frac{1}{\omega}[\epsilon' W_2 \epsilon - \rho_2 \epsilon' W_2' W_2 \epsilon] - \sum_{i=1}^{N} \mu_i/(1 - \rho_2 \mu_i) \qquad [A.5]$$

Differentiating [A.5] with respect to ρ_2 gives:

$$\frac{\partial^2 L}{\partial \rho_2^2} = -\frac{1}{\omega}[\epsilon' W_2' W_2 \epsilon] - \sum_{i=1}^{N} \mu_i^2/(1 - \rho_2 \mu_i)^2 \qquad [A.6]$$

From [A.2]:

$$\frac{\partial L}{\partial \rho_1} = \frac{1}{\omega}[y' W_1' A_2' A_2 (y - \rho_1 W_1 y - X\beta)]$$

$$= \frac{1}{\omega}[y' W_1' A_2' A_2 (A_1 y - X\beta)]$$

$$= \frac{1}{\omega}[y' W_1' A_2' A_2 \epsilon]$$

$$= \frac{1}{\omega}[y' W_1' (I - \rho_2 W_2' - \rho_2 W_2 + \rho_2^2 W_2' W_2)\epsilon]$$

$$\frac{\partial^2 L}{\partial \rho_1 \partial \rho_2} = -\frac{1}{\omega}[y' W_1' (W_2' + W_2 - 2\rho_2 W_2' W_2)\epsilon]$$

$$= -\frac{1}{\omega}[y'W_1'V_2\epsilon] \qquad [A.7]$$

where $V_2 = (W_2' + W_2 - 2\rho_2 W_2'W_2)$

Restating equation [10]:

$$\frac{\partial L}{\partial \omega} = -\frac{N}{2\omega} + \frac{1}{2\omega^2}[\epsilon' A_2' A_2 \epsilon] \qquad [A.8]$$

so

$$\frac{\partial^2 L}{\partial \omega^2} = \frac{N}{2\omega^2} - \frac{2}{2\omega^3}[\epsilon' A_2' A_2 \epsilon]$$

$$= \frac{1}{2\omega^2}\left[N - \frac{N\omega}{2\omega}\right] \text{ (at the minimum) from [11]}$$

$$= -\frac{N}{2\omega^2} \qquad [A.9]$$

From [A.8]:

$$\frac{\partial^2 L}{\partial \omega \partial \rho_2} = \frac{1}{2\omega^2}\frac{\partial}{\partial \rho_2}(\epsilon' A_2' A_2 \epsilon)$$

$$= -\frac{1}{\omega^2}[\epsilon' W_2 \epsilon - \rho_2 \epsilon' W_2' W_2 \epsilon] \qquad [A.10]$$

From [A.8]:

$$\frac{\partial^2 L}{\partial \omega \partial \rho_1} = \frac{1}{2\omega^2}\frac{\partial}{\partial \rho_1}(\epsilon' A_2' A_2 \epsilon)$$

$$= \frac{1}{2\omega^2}[-2y'W_1'A_2'A_2y + 2\rho_1 y'W_1'A_2'A_2W_1y + 2y'W_1'A_2'A_2X\beta]$$

$$= -\frac{1}{\omega^2}[y'W_1'A_2'A_2y - \rho_1 y'W_1'A_2'A_2W_1y - y'W_1'A_2'A_2X\beta]$$

$$= -\frac{1}{\omega^2}[y'W_1'A_2'A_2(I - \rho_1 W_1)y - y'W_1'A_2'A_2X\beta]$$

$$= -\frac{1}{\omega^2}[y'W_1'A_2'A_2\epsilon]$$

$$= -\frac{1}{\omega^2}[\epsilon'A_2'A_2W_1y] \qquad [A.11]$$

Restating [8]:

$$\frac{\partial L}{\partial \beta} = -\frac{1}{\omega}[-X'A_2'A_2A_1y + X'A_2'A_2X\beta] \qquad [A.12]$$

Differentiating with respect to β gives:

$$\frac{\partial^2 L}{\partial \beta^2} = -\frac{1}{\omega}(X'A_2'A_2X) \qquad [A.13]$$

From [A.12]:

$$\frac{\partial^2 L}{\partial \beta \partial \omega} = \frac{1}{2\omega^2}[X'A_2'A_2A_1y - X'A_2'A_2X\beta]$$

But at the maximum value of L(y), $\hat{\beta} = (X'A_2'A_2X)^{-1}X'A_2'A_2A_1y$

so

$$\frac{\partial^2 L}{\partial \beta \partial \omega} = 0 \quad \text{at the minimum} \qquad [A.14]$$

From [A.12]:

$$\frac{\partial^2 L}{\partial \beta \partial \rho_1} = \frac{1}{\omega}\left[X'A_2'A_2\frac{\partial}{\partial \rho}A_1y\right]$$

$$= -\frac{1}{\omega}\,[X'A_2'A_2W_1y] \qquad [A.15]$$

From [A.12]:

$$\frac{\partial^2 L}{\partial\beta\partial\rho_2} = \frac{1}{\omega}\left[X'\,\frac{\partial}{\partial\rho_2}\,(A_2'A_2)A_1y - X'\,\frac{\partial}{\partial\rho_2}\,(A_2'A_2)X\beta\right]$$

$$= -\frac{1}{\omega}\,[X'(W_2' + W_2 - \rho_2W_2'W_2)A_1y$$

$$- X'(W_2' + W_2 - \rho_2W_2'W_2)X\beta]$$

$$= -\frac{1}{\omega}\,[X'(W_2' + W_2 - \rho_2W_2W_2)\,(A_1y - X\beta)]$$

$$= -\frac{1}{\omega}\,[X'V_2\epsilon] \qquad [A.16]$$

Having obtained the $\partial^2L/\partial\beta\partial\rho_2$ we turn now to construct the entries of the matrix $-E[\partial^2L/\partial\theta_r\partial\theta_s]$. From [A.9]:

$$-E\left[\frac{\partial^2 L}{\partial\omega^2}\right] = \frac{N}{2\omega^2} \qquad [A.17]$$

From [A.11]:

$$\frac{\partial^2 L}{\partial\omega\partial\rho_1} = -\frac{1}{\omega^2}\,[(\nu'A_2'^{-1})A_2'A_2W_1(A_1^{-1}A\beta + A_1^{-1}A_2^{-1}\nu)]$$

as $\epsilon = A_2^{-1}\nu$ and $y = A_1^{-1}X\beta + A_1^{-1}A_2^{-1}\nu$. Hence,

$$\frac{\partial^2 L}{\partial\omega\partial\rho_1} = -\frac{1}{\omega^2}\,[\nu'A_2W_1A_1^{-1}A_2^{-1}\nu' + \nu'A_2W_1A_1^{-1}X\beta]$$

and

$$-E\left[\frac{\partial^2 L}{\partial\omega\partial\rho_1}\right] = \frac{1}{\omega^2}\,E[\nu' B_1 \nu] \text{ with } B_1 = A_2 W_1 A_1^{-1} A_2^{-1}$$

$$= \frac{1}{\omega^2}\,E[\mathrm{tr}(\nu' B_1 \nu)]$$

$$= \frac{1}{\omega^2}\,\mathrm{tr}(B_1)\sigma^2 I$$

$$= \frac{1}{\omega}\,\mathrm{tr}(B_1) \qquad [A.18]$$

From [A.10]:

$$\frac{\partial^2 L}{\partial\omega\partial\rho_2} = -\frac{1}{\omega^2}\,[\epsilon'(I - \rho_2 W_2')W_2\epsilon]$$

$$= -\frac{1}{\omega^2}\,[\nu' W_2(A_2^{-1}\nu)]$$

$$-E\left[\frac{\partial^2 L}{\partial\omega\partial\rho_2}\right] = \frac{1}{\omega^2}\,E[\mathrm{tr}\nu' W_2 A_2^{-1}\nu]$$

$$= \frac{1}{\omega^2}\,\mathrm{tr}(B_2)\sigma^2 I \text{ with } B_2 = W_2 A_2^{-1}$$

$$= \frac{1}{\omega}\,\mathrm{tr}(B_2) \qquad [A.19]$$

From [A.14]:

$$-E\left[\frac{\partial^2 L}{\partial\omega\partial\beta}\right] = 0 \qquad [A.20]$$

Write

$$\alpha_1 = -\sum_{i=1}^{N} \lambda_i^2/(1-\rho_1\lambda_i)^2$$

then from [A.3]:

$$\frac{\partial^2 L}{\partial \rho_1} = \alpha_1 - \frac{1}{\omega}\,[(\nu' A_2'^{-1} A_1'^{-1} + \beta' X' A_1'^{-1}) W_1' A_2' A_2 W_1 (A_1^{-1} X\beta + A_1^{-1} A_2^{-1} \nu)]$$

$$-E\left[\frac{\partial^2 L}{\partial \rho_1^2}\right] = \frac{1}{\omega}\,E[\nu' A_2'^{-1} A_1'^{-1} W_1' A_2' A_2 W_1 A_1^{-1} A_2^{-1} \nu + \beta' X' A_1'^{-1} W_1' A_2' A_2 W_1 A_1^{-1} X\beta] - \alpha_1$$

$$= \frac{1}{\omega}\,E[\nu' B_1' B_1 \nu + \beta' X' C X\beta] - \alpha_1$$

where $C = A_1'^{-1} W_1' A_2' A_2 W_1 A_1^{-1}$

Hence,

$$-E\left[\frac{\partial^2 L}{\partial \rho_1^2}\right] = \frac{1}{\omega}\,[E\mathrm{tr}(\nu' B_1' B_1 \nu) + \beta' X' C X\beta] - \alpha_1$$

$$= \frac{1}{\omega}\,[\mathrm{tr}(B_1' B_1)\sigma^2 I + \beta' X' C X\beta] - \alpha_1$$

$$= \frac{1}{\omega}\,[\omega \mathrm{tr}(B_1' B_1) + \beta' X' C X\beta] - \alpha_1$$

$$= \mathrm{tr}(B_1' B_1) - \alpha_1 + \frac{1}{\omega}\,\beta' X' C X\beta \qquad \text{[A.21]}$$

From [A.7]:

$$\frac{\partial^2 L}{\partial \rho_1 \rho_2} = -\frac{1}{\omega}\,[y'W_1'V_2\epsilon]$$

$$= -\frac{1}{\omega}\,[(\nu' A_2'^{-1} A_1'^{-1} + \beta' X' A_1'^{-1}) W_1' V_2 A_2^{-1}\nu]$$

So

$$-E\left[\frac{\partial^2 L}{\partial \rho_1 \rho_2}\right] = \frac{1}{\omega}\,E[\nu' A_2'^{-1} A_1'^{-1} W_1 V_2 A_2^{-1}\nu]$$

$$= \frac{1}{\omega}\,E[\nu' D\nu] \quad \text{with } D = A_2'^{-1} A_1'^{-1} W_1 V_2 A_2^{-1}$$

$$= \frac{1}{\omega}\,E[\mathrm{tr}\,\nu' D\nu]$$

$$= \frac{1}{\omega}\,\mathrm{tr}(D)\sigma^2 I = \mathrm{tr}(D) \qquad [A.22]$$

From [A.15]:

$$\frac{\partial^2 L}{\partial \beta \partial \rho_1} = -\frac{1}{\omega}\,[X' A_2' A_2 W_1 (A_1^{-1} X\beta + A_1^{-1} A_2^{-1}\nu)]$$

So

$$-E\left[\frac{\partial^2 L}{\partial \beta \partial \rho_1}\right] = \frac{1}{\omega}\,[X' A_2' A_2 W_1 A_1^{-1} X\beta] \qquad [A.23]$$

From [A.6] with

$$\alpha_2 = -\sum_{i=1}^{N} \mu_i^2/(1-\rho_2\mu_i)^2$$

$$\frac{\partial^2 L}{\partial \rho_2^2} = \alpha_2 - \frac{1}{\omega}\,[\nu' A_2'^{-1} W_2' W_2 A_2^{-1}\nu]$$

Hence

$$-E\left[\frac{\partial^2 L}{\partial \rho_2^2}\right] = \frac{1}{\omega}\, E[\mathrm{tr}(\nu' B_2' B_2 \nu)] - \alpha_2$$

$$= \frac{1}{\omega}\, \mathrm{tr}(B_2' B_2)\sigma^2 I - \alpha_2$$

$$= \mathrm{tr}(B_2' B_2) - \alpha_2 \qquad [A.24]$$

From [A.16]:

$$\frac{\partial^2 L}{\partial \beta \partial \rho_2} = -\frac{1}{\omega}\, [X' V_2 A_2^{-1} \nu]$$

Hence

$$-E\left[\frac{\partial^2 L}{\partial \beta \partial \rho_2}\right] = 0 \quad \text{as a column vector} \qquad [A.25]$$

From [A.13]:

$$-E\left[\frac{\partial^2 L}{\partial \beta^2}\right] = \frac{1}{\omega}\, X' A_2' A_2 X \qquad [A.26]$$

Thus the variance-covariance matrix of the coefficient estimates is given in [A.26] where:

$$B_1 = A_2 W_1 A_1^{-1} A_2^{-1}$$

$$B_2 = W_2 A_2^{-1}$$

$$C = A_1'^{-1} W_1' A_2' A_2 W_1 A_1^{-1}$$

$$D = A_2'^{-1} A_1'^{-1} W_1 V_2 A_2^{-1}$$

$$V_2 = W_2' + W_2 - 2\rho_2 W_2' W_2$$

$V(\hat{\omega}, \hat{\rho}_1, \hat{\rho}_2, \hat{\beta}) =$

$$\omega^2 \begin{bmatrix} N/2 & \omega tr(B_1) & \omega tr(B_2) & 0' \\ \omega tr(B_1) & \omega^2(trB_1'B_1 - \alpha_1) + \omega\beta'X'CX\beta & \omega^2 tr(D) & (\omega X'A_2'A_2W_1A_1^{-1}X\beta)' \\ \omega tr(B_2) & \omega^2 tr(D) & \omega^2(trB_2'B_2 - \alpha_2) & 0' \\ 0 & \omega X'A_2'A_2W_1A_1^{-1}X\beta & 0 & \omega X'A_2'A_2X \end{bmatrix}^{-1}$$

NOTES

1. It is convenient to have W_1 and W_2 with row sums of unity (Ord, 1975) in which case the maximum eigenvalue is 1. It can be shown (Doreian and Hummon, 1976: 128; Ord, 1975: 121) that $\rho < 1/\max(\lambda)$, and so both spatial parameters are bounded above by 1.

2. Throughout, the convention is used where Y refers to the dependent variable and y to an observed distribution for that variable.

3. Henceforth L will denote the log-likelihood function and not the likelihood function.

4. The alternative strategy concentrates on the log-likelihood function and uses a search procedure. The log-likelihood function [7c] can be reexpressed as:

$$L(y, \rho, \rho_1, \hat{\beta}, \hat{\omega}) = \text{const} - \left(\frac{N}{2}\right)\ln\hat{\omega} + \ln|A_1| + \ln|A_2|$$

and the estimation task is to find the combination of ρ_1 and ρ_2 that maximizes this equation or, equivalently, minimizes:

$$\ln\hat{\omega} - \left(\frac{N}{2}\right)\ln|A_1| - \left(\frac{N}{2}\right)\ln|A_2|$$

Let $\{\lambda_i\}$ be the eigenvalues of W_1 and $\{\mu_i\}$ the eigenvalues of W_2, then (Ord, 1975):

$$|A_1| = \prod_{i=1}^{N} (1 - \rho_1\lambda_i)$$

$$|A_2| = \prod_{i=1}^{N} (1 - \rho_2\mu_i)$$

From [12] it is straightforward to show, using [9], that:

$$N\hat{\omega} = y'A_1'A_2'[I - A_2X\{(A_2X)'(A_2X)\}^{-1}(A_2X)']A_2A_1y$$

Defining $M_2 = I - A_2X\{(A_2X)'(A_2X)\}^{-1}(A_2X)'$, we have

$$N\hat{\omega} = y'A_1'A_2'M_2A_2A_1y$$

From these equations we seek the combination of ρ_1 and ρ_2 that minimizes:

$$\ln(y'A_1'A_2'M_2A_2A_1y) - \frac{2}{N}\sum_{i=1}^{N}\ln(1-\rho_1\lambda_i) - \frac{2}{N}\sum_{i=1}^{N}\ln(1-\rho_2\mu_i)$$

This is done by a direct search procedure on ρ_1 and ρ_2 simultaneously. Knowing that $1 > \rho_i$, this expression is computed for a set of values of ρ_1 and ρ_2. A coarse search using increments of 0.1 for each of ρ_1 and ρ_2 can locate the minimum roughly, and then smaller increments can be used through a restricted range around this rough solution.

5. From the appendix:

$$\frac{\partial L}{\partial \rho_1} = \left(\frac{1}{\omega}\right)[y'W_1'A_2'A_2y - \rho_1 y'W_1'A_2'A_2W_1y - y'W_1'A_2'A_2X\beta]$$

$$- \sum_{i=1}^{N}\lambda_i/(1-\rho_1\lambda_i)$$

and

$$\frac{\partial L}{\partial \rho_2} = \left(\frac{1}{\omega}\right)[\epsilon'W_2\epsilon - \rho_2\epsilon'W_2'W_2\epsilon] - \sum_{i=1}^{N}\mu_i/(1-\rho_2\mu_i)$$

From the estimating equations:

$$\hat{\beta} = (X'A_2'A_2X)^{-1}X'A_2'A_2A_1y$$

$$\hat{\epsilon} = A_1y - X\hat{\beta}$$

and $$\hat{\omega} = (1/N)\epsilon'A_2'A_2\epsilon$$

These equations are substituted into $\partial L/\partial\rho_1 = 0$ and $\partial L/\partial\rho_2 = 0$ to give the nonlinear equations requiring solution.

6. An efficient computer program for doing this (as part of a general spatial analysis package) is being written at the Social Science Computer Research Institute at the University of Pittsburgh and will be made available.

7. The examples reported here are each based on a single body of data. A full Monte Carlo study will clarify the essential properties of these estimators with respect to bias, if any, and sample variability. Such a study is currently being designed.

8. The values are suggested by the analysis of Doreian (1981: Table 2). Note that one parameter is set to 0.

9. The full set of categories are Urban, South Louisiana plantation, South Louisiana bayou, Florida, South West Louisiana, North Louisiana plantation, Macon Ridge, North Louisiana Hills and Central Pine Hills (see Howard, 1971 for details).

10. The ratio of $\hat{\beta}$ to its corresponding estimated standard error is distributed as a t-statistic and a 2-tail test considered.

11. Of course, in many situations the precise value of the intercept will not be of interest. Compare Rao and Miller's (1971: 5-6) amusing argument on this. If the intercept is interpreted as the mean impact of excluded exogenous variables, this becomes a more important issue.

12. That such a ceiling exists is not surprising nor is it conceptually bothersome. In the estimation procedure for the parameter estimates and the variance-covariance matrix for these estimates $(I - \rho_1 W_1)^{-1}$ and $(I - \rho_2 W_2)^{-1}$ are involved. Given that $(I - \rho W)^{-1} = I + \rho W + \rho^2 W^2 + \rho^3 W^3 + \ldots$ it is clear that this converges far less rapidly for high ρ. This is likely to be the technical reason for the poor performance of the estimation procedure for high ρs. When the specification of the model is considered, a statement that $Y = WY + X\beta + \epsilon$ is one saying there are subareas of great homogeneity where the value of one area is the same as that in adjacent areas. If this applies throughout the region, then Y is essentially the same throughout the region. With no variability to account for, there is little point in specifying a model.

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