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A partitioning approach to structural balance

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Abstract

The classic formulation of structural balance by Cartwright and Harary (Psychological Review, 63, 1956, 277–293) had the basic structural theorem that a balanced structure could be partitioned into two mutually antagonistic subgroups each having internal solidarity. Davis (Human Relations, 20, 1967, 181–187) extended this theorem for cases where there can be more than two such mutually antagonistic subgroups. We use these theorems to construct a criterion function for a local optimization partitioning procedure for signed digraphs. For any signed digraph, this procedure yields those partitions with the smallest number of errors, a measure of the imbalance in the graph, and an identification of those links inconsistent with both generalized and structural balance. These methods are applied to some artificial data and to the affect data from Sampson (A novitiate in a period of change: An experimental and case study of social relationships, Dissertation, Cornell University, 1968). The latter provides a positive test of a basic tenet of balance theory, that there is a tendency towards balance with signed relations in human groups. While these methods can be applied to all signed digraphs and signed graphs, the balance hypothesis is relevant only for affect ties.

1. Introduction

Structural balance theory has been an enduring concern for many social psychologists and sociologists. Our initial focus here is on the set of theorems in the formalized version of structural balance theory, particularly the basic structure theorem of Cartwright and Harary (1956). We use these theorems as the foundation for a set of procedures that, for a signed social network, (i) establish those partitions having an optimized criterion function as close as possible to balance, (ii) establish a measure of the extent to which a structure is imbalanced and (iii)

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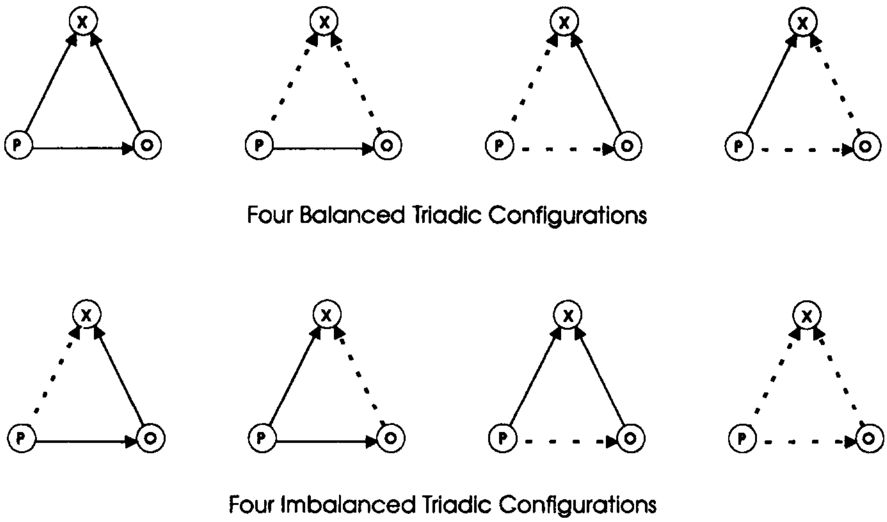

Fig. 1. Eight directed POX triads.

identify those ties (and, by implication, those actors) that contribute to a lack of balance in a social structure represented in network terms.

Heider (1946, 1958) is credited with providing the initial *systematic* statement of structural balance theory (Taylor, 1970). In his formulation, there are three objects: a focal person, P, another actor, O, and a nonperson object, X. The object, X, can be "a situation, an event, an idea or a thing" (Heider, 1946 p. 107). For P, these entities form a coherent whole with sentiment relations between P and O and unit relations between the actors and X. The theoretical unit, POX, is a triad that can take any of the basic forms shown in Fig. 1, where solid lines represent positive ties and dashed lines represent negative ties.

A triad is (sign-) balanced if the product of the signs in the triad is positive and it is (sign-) imbalanced if the product of the signs is negative [1]. The top panel of Fig. 1 displays the balanced triads with imbalanced triads in the lower panel. In Heider's theory, people prefer balanced triads to those that are imbalanced. Given this, balanced triads are stable, in the theory, while imbalanced triads generate tension and forces towards balance.

The same ideas of tensions and forces are present as key ideas of Newcomb's (1956, 1961) statement of structural balance theory. For his POX triad, X can be a nonperson object, another actor in the group or even the entire group [2]. Heider's formulation deals only with signed networks (where the value of the ties are 1, 0 or

[1] Henceforth we refer simply to balance and imbalance.

[2] Newcomb builds in additional structural elements, such as P's perception of O's sentiment towards P, in addition to O's sentiment towards P. These can be incorporated straightforwardly into the formal models stemming from the work of Cartwright and Harary (1956).

-1), a strategy used also by Cartwright and Harary (1956). Newcomb deals with tie strengths. A network can be signed-balanced but not balanced if, for example, P likes O more than O likes P. The procedures outlined below can handle both binary signed graphs and valued signed graphs.

2. Balanced signed digraphs

2.1. Signed graphs

A signed digraph [3] is an ordered pair, (G, σ), where (Batagelj, 1994, p. 56):

(i) $G = (V, R)$ is a digraph, without loops, having a set of vertices, V, and a set of arcs, $R \sqsubseteq V \times V$; and

(ii) σ: $R \rightarrow \{p, n\}$ is a sign function. The arcs with the sign p are positive while the arcs with the sign n are negative. Equivalently, and consistent with most diagrams of signed graphs; σ: $R \rightarrow \{+1, -1\}$.

As a shorthand, we denote such a graph (V, R, σ). For the POX triads of Heider and of Newcomb, {P, O, X} ϵV while the relations among them ϵR. For a social network, the actors $\{v_i\}$ ϵV and ordered pairs $(v_i, v_j) \epsilon R$ for $v_i, v_j \epsilon V$. The signs of paths, cycles, semipaths and semicycles are determined by the product of the signs of the arcs contained in them. They are either positive or negative and a digraph, G, is balanced if every semicycle in G is positive.

2.2. Structural balance in signed graphs

The following partitioning approach to structural balance is based on the well-known results from Harary et al. (1965, p. 342).

Theorem 1. A signed digraph, $G = (V, R, \sigma)$, is balanced if and only if, for every pair of vertices in V, all semipaths joining them have the same sign.

Theorem 2. A signed digraph is balanced if and only if every semicycle is positive.

Theorem 3. A signed digraph, $G = (V, R, \sigma)$ is balanced if and only if V can be partitioned in two subsets A and B such that:

(i) $\forall(v_i, v_j)$ *mapped to p under* σ *either* $v_i, v_j \epsilon A$ *or* $v_i, v_j \epsilon B$, and

(ii) $\forall(v_i, v_j)$ *mapped to n under* σ, $v_i \epsilon A$ *and* $v_j \epsilon B$, *or vice versa.*

See also Roberts (1976, p. 70). Theorem 3 has been called the 'structure theorem', for obvious reasons. A (macro-) network property is one where there are *two* mutually antagonistic subgroups [4], each of which has internal solidarity (only positive ties).

[3] These definitions can be restricted straightforwardly to graphs with undirected edges. Indeed, the presentation of Cartwright and Harary (1956) used such graphs.

[4] We ignore the case where one of the two subgroups in the partition is empty. Additionally, we assume the digraph is not disconnected.

2.3. Partitionable signed graphs

Davis (1967) noted that social groups/networks frequently have multiple coalitions with negative ties between coalitions and sought the conditions under which such partitions could occur. The all negative triad in Fig. 1 is ambiguous and need not be defined as imbalanced. Indeed, when it is defined as balanced, two items change. Instead of examining the sign of a semicycle we examine whether or not it contains a *single* negative arc. Second, instead of a partition into two subsets we look at partitions into more than two subsets. A signed digraph, G, is k-balanced if V can be partitioned into k subsets, called "plus-sets", such that the positive ties are found only within plus-sets and the negative ties go between plus-sets [5]. Any signed graph that is not k-balanced is k-imbalanced for $k \geq 2$.

Theorem 4. A signed diagraph is k-balanced if and only if it contains no semicycles with exactly one negative arc (Davis, 1967, p. 181).

We reserve the term 'balance' for $k = 2$, its more traditional usage, and use the term 'generalized balance' for $k > 2$.

2.4. Measures of generalized imbalance

The initial psychological and social psychological theories posited a tendency towards balanced structures. Empirically, if a signed network is changing through time towards balance (or not), it becomes necessary to measure the extent to which this network is imbalanced in order to test the theoretical hypothesis of movement towards balanced structures. Two broad classes of measures of balance have been proposed (see Harary et al., 1965, pp. 346–352). The first uses the relative counts of balanced semicycles to all of the semicycles in the graph. The other uses the negation of the arcs that contribute to imbalance. An arc index of imbalance in a signed digraph is the number of arcs that must be changed in sign in order to construct a balanced network. More precisely, a collection of arcs "is called negation-minimal if its negation results in balance, but the negation of any proper subset of (the set) does not" (Harary et al., 1965, p. 349) [6]. Henceforth, we will use the negation-minimal set of arcs as a measure of imbalance in a signed diagraph. The concept of negational-minimal sets generalizes straightforwardly to the cases of $k > 2$.

[5] Davis (1967) used the term 'clustering' to distinguish the phenomenon of multiple clusters from the balance partition into clusters. We subsume both balanced and clusterable under the term k-balanced and reserve clustering and clusterable for the broad area of cluster analysis.

[6] An alternative arc index is the number of arcs that must be removed in order to establish a balanced structure. A deletion-minimal set of arcs can be specified in a similar fashion. However Harary et al. (1965, p. 350) prove that any deletion-minimal set of arcs of a signed digraph is negation-minimal and conversely. This applies also for $k > 2$.

3. Partitioning via optimizational techniques

Ferligoj et al. (1994) have advocated an approach to block model partitioning via local optimization procedures. For structural equivalence (Batagelj et al., 1992a), for regular equivalence (Batagelj et al., 1992b) and generalized concepts of equivalence (Doreian et al., 1994), a criterion function is specified and minimized. For a specific equivalence type, the optimizational problem is: determine the clustering (partition) C^* for which:

$$P(C^*) = \min_{C \in \Phi} P(C)$$

where C is a clustering of the given set of entities, V, Φ is the set of all possible clusterings and P: $\Phi \to \mathscr{R}$, the range of the criterion function.

The original structure theorem (Theorem 3) and its generalization (Theorem 4) specify partitions of V. Each partition has plus-sets and a distribution of ties that conform to the type of partition considered. The negation-minimal index of imbalance and generalized imbalance suggests a natural criterion function. Departures from structural or generalized balance are either negative arcs within plus-sets or positive arcs between plus-sets. Each can be viewed as a type of error and the natural criterion function is:

$$P(C) = \sum\nolimits_{\mathrm{n}} + \sum\nolimits_{\mathrm{p}} \tag{1}$$

where Σ_{p} is the number of positive arcs between plus-sets and Σ_{n} is the number of negative arcs within plus-sets. Implictly, these types of error are equally important. However, in some empirical context, one type of error may be more important than the other. For example, negative arcs within plus-sets may generate greater tension than positive arcs between plus-sets. The criterion function can be rewritten as:

$$P(C) = \alpha\sum\nolimits_{\mathrm{n}} + (1-\alpha)\sum\nolimits_{\mathrm{p}} \tag{2}$$

where $0 \le \alpha \le 1$. If $\alpha = 0.5$, the types of error are equally important and (2) is merely a reweighting of (1). For $0 \le \alpha < 0.5$, positive errors are more consequential, while for $0.5 < \alpha \le 1.0$, the negative errors are more important. Herein, the types of errors are treated as equally important.

We propose the following local optimization procedure. Let k be the number of plus-sets for which a partitioning is sought. The set, V, is randomly partitioned into k clusters and the partition has a set of neighbors. These neighbors are obtained from the clustering either by moving one vertex from one cluster to another cluster or by interchanging two vertices from different clusters. The criterion function is computed for the initial random partition. Neighbors of a given solution are examined and the criterion function is computed for the neighbors. If a partition with a lower value of the criterion function is found we move to that solution. If not, we examine another neighbor of the current partition. Note that the method of determining neighbors is a random selection. Further, if interchanges are implemented, we do not examine all possible interchanges but make a random selection from a randomly generated set of possible interchanges. The movement of one vertex from a cluster to another is also examined randomly. When we compute the value of the criterion function for a selected neighbor, we do not need to compute the criterion function for the whole graph. Such a computation would have a time complexity of $O(n^2)$. Instead, we compute the difference in the criterion function, a computation whose time complexity is only $O(n)$. This procedure is repeated many times. From each random partition into k clusters, the minimal value of the criterion function is computed and the set of final partitions are those having the minimized value of the criterion function over all of the random starting partitions.

This procedure is one where the number of clusters is specified and a local optimization routine is used to locate the set of partitions having the minimum value of the criterion function. These clusters will be as close as possible to the corresponding plus-sets of the partition. In addition to having these partitions, we have also the arc index of the considered form of balance, either derived from Eq. (1) or Eq. (2), and we have the identification of the arcs that are not consistent with the underlying form of balance of the partitioned structure.

As the algorithm is the implementation of a local optimization procedure, there is the risk of obtaining only local optima. For small graphs, it is possible to search through all partitions. When we have done this we know the global minimum of the criterion function. In these cases, the local optimization procedure always located the global minimum. We emphasize the need to repeat the procedure many times [7]. One perhaps surprising feature is that it is possible to have multiple partitions, for a given value of k, with the same minimum value of the criterion function. We note that Davis (1967, p. 183) anticipated this for $k > 2$, and it can hold also for $k = 2$.

We doubt that it is useful to specify a specific partition as a starting point for the local optimization procedure. The obvious candidates are partitions from some structural equivalence algorithms, for example, CONCOR. Doreian et al. (1995) show that structural equivalence partitions of signed graphs (with stacked positive and negative ties) lead to partitions with very high criterion functions and partitions far from k-balanced partitions [8]. The local optimization procedure used herein is very fast and, we believe, reaches partitions with lower values of the criterion function [9].

[7] For the Sampson 18 person example (with one relation) considered below, a set of 12 repetitions, with between 5500 and 7800 relocations takes about 6 s on a 486 machine running at 66 MHz. The number of relocations increases with k but increases the running time by, at most, 1 s. Thus far, we have not had to repeat the entire process more than 15 times. Nor is it necessary to increases the number of repetitions above 12, as, in most cases, the optimum value of $P(C^*)$ has been reached by then.

[8] Doreian et al. (1994) note similar, but far less extreme phenomena, when attention is confined to structural equivalence partitions of a social relation.

[9] In our experience, constructing, preparing a prior partition and using it as input to the partitioning algorithm, takes longer than a full implementation of the current procedure.

4. Illustrative examples

To mobilize the optimization procedure advocated here, two parameters are specified, namely k and α. For structural balance, the only value of k is 2, while for the generalized balance of Davis (1967), k is greater than 2. For our purposes, we deal with $k \geq 2$. Fig. 2 contains a signed digraph with nine vertices. In general, a relatively small number of values of k are used but, for this example, we examine all possible values of k.

Table 1 shows the value of the criterion function (from Eq. (2)) for each number of clusters together with the number of partitions that have the minimized value of the criterion function. At one extreme, having all of the vertices in a single cluster means that all of the negative arcs in the graph contribute to the error score. At the other extreme, having a partition of the graph into singletons means that all of the positive arcs contribute to the criterion function [10]. The highest values are at these extremes. In general, we have found that the graph of the values of the criterion function against the number of clusters has a general concave upwards shape implied by the values of Table 1.

According to those values, there is a unique partition into two clusters having the minimized value of the criterion function. This partition is: {a, e, f, i}, {b, c, d, g, h} and the arcs that are inconsistent with structural balance are the negative a to e link and the negative h to g link. Both of these errors are found in imbalanced cycles in this graph. Similarly, the unique partition of the vertices into three clusters is: {a, e, f}, {b, c, d, g, h}, {i}. This three-cluster solution identifies the same two arcs as being inconsistent with generalized balance.

Partitions of the vertices into four, five or six clusters each yields more than one partition with the minimized value of the criterion function. In general, we are inclined to accept the partition with the *minimized* overall value of the criterion function and we prefer either a unique partition corresponding to such a value of the criterion function, or having a *small* number of partitions each having the same value of the criterion function.

We turn now to examine the two graphs used by Davis (1967) to illustrate the generalized balance approach. These are shown in Fig. 3. The graph on the left is k-balanced (with zero error) while the other is not. Table 2 shows the values of the criterion functions for each graph and all numbers of clusters in partitions.

The k-balanced graph has two partitions with zero errors. The clustering {a, c}, {b}, {d, e, f} is three-balanced. The partition into four clusters where a, b and c are singletons, with {d, e, f} as the fourth cluster, is four-balanced. These are exactly the partitions identified by Davis. The second panel of Table 2 reveals that the second graph of Fig. 3 cannot be partitioned so as to have a criterion function of zero regardless of the number of clusters, as pointed out by Davis.

[10] There are ten positive arcs in the graph and with $\alpha = 0.5$, the value of the criterion function expressed in Eq. (2) is 5. Similarly, there are six negative arcs in the graph which means that Eq. (2) yields an error score of 3.

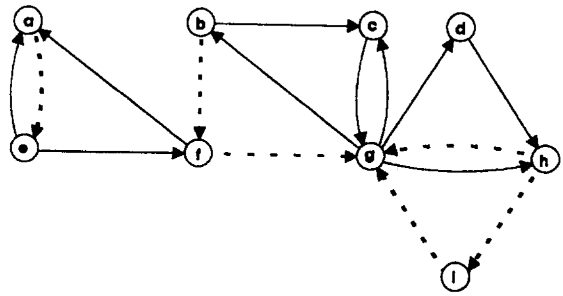

Fig. 2. A 9-node signed digraph (Source: Harary et. al., 1965, p. 347).

Table 1
Criterion function values for Fig. 1

Number of clusters	Minimized value of criterion function	Number of partitions
1	3.0	1
2	1.0	1
3	1.0	1
4	1.5	3
5	2.0	6
6	2.5	4
7	3.0	1
8	4.0	1
9	5.0	1

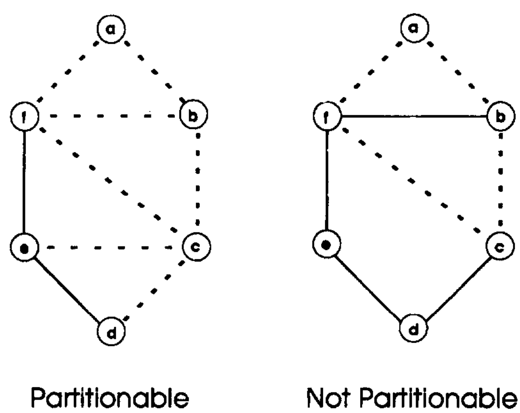

Fig. 3. Signed graphs (Source: Davis, 1967, p. 182).

Table 2
Criterion function values for Fig. 2

Number of clusters	Minimized value of criterion function	Number of partitions
The k-balanced graph		
1	7	1
2	1	1
3	0	1
4	0	1
5	1	2
6	2	1
The k-imbalanced graph		
1	4	1
2	1	3
3	1	3
4	2	6
5	3	4
6	4	1

In discussing measures of structural balance, Harary et al. (1965, pp. 348–349) present an analysis of a stage play having four scenes ordered in time. Fig. 4 displays the signed relations among the four characters in each of the play's scenes.

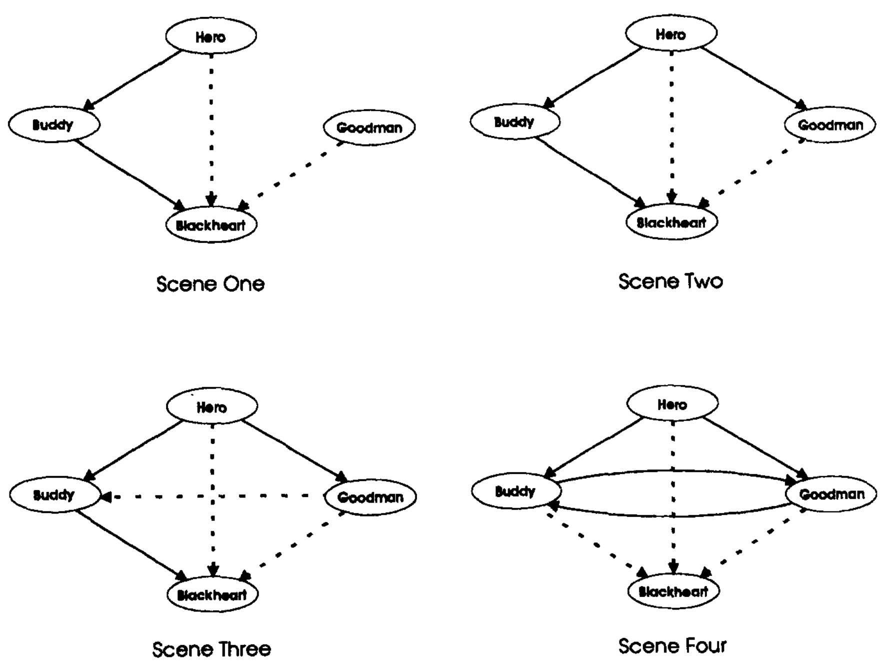

Fig. 4. Four play scenes.

Table 3
Criterion function values for Fig. 3

Number of clusters	Minimized value of the criterion function			
	Scene one	Scene two	Scene three	Scene four
1	1.0	1.0	1.5	1.5
2	0.5	0.5	0.5	0
3	0.5	1.0	1.0	1.0
4	1.0	1.5	1.5	2.0

The play was written to show how the relations changed through time and that at the conclusion of the play the structure was balanced. Table 3 shows the value of the criterion function for each of the scenes depicted in Fig. 4. Given the focus on structural balance, the appropriate number of clusters is 2 and the second row of Table 3 provides the criterion function (using Eq. (2)) as a measure of balance. It is clear that over the first three time points, the structure is not balanced but, at the conclusion of the play, the structural configuration is balanced.

Harary et al. (1965) also suggest that the line index of balance [11] can be used to identify those links in a network that will change to result in a balanced structure. This hope is unlikely to be fulfilled. Consider the graph at Scene One from Fig. 4. There are three distinct partitions of these vertices that have the same lowest value of the criterion function. One is {Hero, Buddy, Blackheart} and {Goodman}. The link identified as inconsistent with balance is the negative link from Hero to Blackheart. However, if either of the positive links in the three person triad become negative, the overall structure is balanced. Note also that the Hero — Blackheart link remains negative in each scene, including the final balanced structure. Furthermore, the partitions do not uniquely identify the arc requiring change. Another partition with a minimal value for the criterion function is {Hero, Goodman} and {Buddy, Blackheart}. For this partition, the link identified as being discrepant from balance is the positive link from Hero to Buddy. The third partition having the minimal criterion function is {Hero, Buddy, Goodman} and {Blackheart}. This time, the link identified as a contributor to the lack of balance is the positive tie from Buddy to Blackheart. Thus, each link in the imbalanced triad can be identified as a link contributing to imbalance depending on the particular partition selected. The set of partitions of the structural configuration, by themselves, are not helpful in discerning *which* link will actually change.

Similar remarks apply to each of the scenes depicted in Fig. 4. If the structure of Scene Three is examined, the partition yielding the minimized score of the criterion function is unique: {Hero, Goodman} and {Buddy, Blackheart}. For this partition, the link identified as inconsistent with balance is the positive link from Hero to Buddy. As can be seen in the final diagram of Fig. 4, this identified link did not change. Thus even for unique partitions, it appears that while the

[11] Use of Eq. (1) gives their index. Our measure in (2) is simply a linear transform of it.

partitioning approach can identify the partition closest to balance and provide the appropriate measure of imbalance, it cannot identify the specific lines that will change in a structure.

We turn now to consider a sustained empirical example.

5. A balance-partitioning analysis of the Sampson monastery data

The Sampson (1968) study provides a rich resource of sociometric data for a variety of social relationships among a group of young men who were either postulants or novices at a New England monastery. We consider the affect relationship for three of the time periods delineated by Sampson. At time T_1, Sampson identified 13 trainee monks of which six remained at time T_2. Among these actors, Peter was a leader with Bonaventure and Berthold joining him as the core of the remaining trainees. Mark is identified as an isolate, with Victor and Ambrose described as peripheral members of the group remaining after T_1. At T_2, 12 members of the new class were admitted to the monastery: John Bosco, Gregory, Basil, Romuald, Louis, Winfred, Amand, Hugh, Boniface, Albert, Elias and Simplicius. Among the new group, John Bosco and Gregory emerged early as leaders and Basil was seen as having a personality problem. Sampson indicates that this actor was tolerated but not accepted by the rest of the group. Additionally, Elias and Simplicius were identified as immature, childish and in need of help. The next time point identified by Sampson, T_3, was described as a period of differentiation and polarization and T_4 was identified as a subsequent time point, one that occurred just before the expulsion of several members of the new class.

Describing the subgroup structure of these actors, Sampson identified three distinct groups. The Young Turks were made up of John Bosco, Gregory, Mark, Winfred, Hugh, Boniface and Albert. The *Loyal Opposition* was comprised of Peter, Bonaventure, Berthold, Victor, Ambrose, Romuald, Louis and Amand. The remaining actors, Basil, Elias and Simplicius, were labelled as the *Outcasts*. This partition is exactly the one reported by Breiger et al. (1975) from an analysis of the T_4 data (for multiple relations).

If the interval from T_2 to T_4 was a period of differentiation and polarization, it is reasonable to expect that, with the passage of time, the (k-) imbalance measure for this group of actors would decrease and that the actors could be partitioned into differentiated subgroups that were mutually antagonistic. Specifically, the partitioning algorithms that we have described herein should yield the coherent subgroups identified by others and the measure of imbalance should decline from T_2 through to T_4.

We present two sets of analyses. Sampson reported his sociometric data in the form of valued graphs. Each actor was asked to provide a list of the three other actors for whom they had the greatest positive affect as well as the three actors they disliked the most. The positive affect ties have values 3, 2 and 1 in decreasing order of positive affect, while -1, -2 and -3 indicate an increased ordering of dislike. The three valued graphs are reported in matrix form in Table 4.

Most discussions of structural balance are restricted to signed *binary* digraphs. The binarized form corresponding to the digraphs represented in Table 4 will be considered later. Before doing so, we consider the graphs represented in Table 4.

To deal with the valued and signed digraphs we need to modify the criterion function to take into account the magnitudes of the ties. Let C be a partition of V into K clusters. Consistent with the Davis (1967) formulation, we call $\{C_k : 1 \leq k \leq K\}$ a set of plus-sets. Let l and m be two points within the plus-set C_k. There will

Table 4
Affect ties among 18 actors at three time points

	T_2																	
1 John Bosco	0	0	2	0	3	−2	−1	0	0	−3	0	0	0	1	0	0	0	0
2 Gregory	3	0	0	0	0	0	2	0	0	−1	0	0	−3	1	0	0	−2	0
3 Basil	2	3	0	−1	0	0	0	−3	−2	0	0	0	0	0	0	0	1	0
4 Peter	0	0	−2	0	3	1	−3	0	0	2	0	0	0	0	0	0	0	−1
5 Bonaventure	0	0	0	3	0	0	0	0	0	0	2	0	1	0	0	0	0	0
6 Berthold	1	0	0	3	0	0	−3	−1	2	0	0	−2	0	0	0	0	0	0
7 Mark	0	2	0	−3	−1	−2	0	1	0	0	0	0	0	0	0	3	0	0
8 Victor	3	2	−3	0	0	0	0	0	1	0	0	0	0	−2	0	0	−1	0
9 Ambrose	0	0	−3	0	2	0	0	3	0	0	0	0	0	0	0	1	−2	−1
10 Romuald	0	0	0	3	0	0	0	1	0	0	0	0	0	2	0	0	0	0
11 Louis	0	0	−1	0	3	0	0	1	0	0	0	0	−3	2	0	0	−2	0
12 Winfrid	3	2	−1	−3	0	0	0	0	0	0	−2	0	0	1	0	0	0	0
13 Amand	0	−3	0	0	2	−2	1	0	0	0	0	−1	0	0	0	0	0	3
14 Hugh	3	0	0	0	0	0	0	−2	0	0	1	2	−3	0	2	0	−1	0
15 Boniface	3	2	−2	0	0	0	0	0	0	0	0	0	−3	1	0	0	−1	−1
16 Albert	1	2	0	0	0	0	3	0	0	0	0	0	−1	0	0	0	−3	−2
17 Elias	0	0	3	−3	−2	0	0	0	0	0	−1	0	2	0	0	0	0	1
18 Simplicius	2	3	0	−3	0	−2	1	0	0	0	0	0	0	0	0	−1	0	0
	T_3																	
1 John Bosco	0	2	0	−2	1	0	0	3	0	−3	−1	0	0	0	0	0	0	0
2 Gregory	3	0	0	−3	0	0	1	−2	0	0	0	2	−1	0	1	0	0	0
3 Basil	3	−1	0	0	0	−2	0	−3	0	0	0	0	0	0	0	0	1	2
4 Peter	−2	−3	0	0	3	2	0	0	0	0	1	0	0	0	0	0	0	0
5 Bonaventure	2	0	0	3	0	0	0	0	0	0	1	0	0	0	0	0	0	0
6 Berthold	1	0	−3	3	0	0	0	−1	2	0	0	−2	0	0	0	0	0	0
7 Mark	1	2	0	−3	0	−2	0	−1	0	0	0	0	0	0	0	3	0	0
8 Victor	−2	−3	−1	3	0	1	0	0	0	2	0	0	0	0	0	0	0	0
9 Ambrose	0	0	−3	0	1	0	0	3	0	0	0	2	0	0	0	0	−2	−1
10 Romuald	0	0	0	3	1	0	0	0	0	0	0	0	2	0	0	0	0	0
11 Louis	−1	−3	−2	2	3	0	0	0	0	0	0	0	0	0	0	1	0	0
12 Winfrid	3	1	−3	0	0	−2	0	0	0	0	0	0	0	2	0	0	−1	0
13 Amand	0	−3	0	0	2	−2	1	0	0	0	0	−1	0	0	0	0	0	3
14 Hugh	3	1	0	−3	0	0	0	0	0	−2	0	1	0	0	2	0	−1	0
15 Boniface	2	3	−2	0	1	0	0	0	0	−1	0	0	−3	1	0	0	−1	0
16 Albert	0	2	−1	−3	0	0	3	0	0	0	0	1	0	0	0	0	−2	0
17 Elias	0	1	2	−2	0	−1	0	0	0	0	−3	0	0	0	0	0	0	3
18 Simplicius	0	1	0	−3	0	−2	0	0	0	0	−1	0	2	0	0	0	3	0

Table 4 (continued)

	T_4																	
1 John Bosco	0	−2	3	0	0	0	−3	0	0	−1	0	1	0	2	0	0	0	0
2 Gregory	3	0	0	−3	0	0	1	−2	0	0	0	2	−1	0	0	0	0	0
3 Basil	3	−2	0	−3	0	−2	0	0	0	0	0	0	2	0	0	−1	1	2
4 Peter	−2	−3	0	0	3	1	0	0	0	0	2	0	0	−1	0	0	0	0
5 Bonaventure	0	0	0	3	0	0	0	0	1	0	2	0	0	0	0	0	0	0
6 Berthold	0	−1	−3	3	1	0	−2	0	2	0	0	0	0	0	0	0	−2	0
7 Mark	0	3	0	−3	0	−2	0	−1	0	0	0	1	0	0	0	2	0	0
8 Victor	0	−3	−2	3	0	2	0	0	1	0	0	0	0	−1	0	0	0	0
9 Ambrose	0	0	−3	0	1	0	0	3	0	0	0	2	0	0	0	0	−2	−1
10 Romuald	0	0	0	3	1	0	0	0	1	0	0	0	2	0	0	0	0	0
11 Louis	−1	−3	−2	0	2	0	0	3	0	0	0	0	0	1	0	0	0	0
12 Winfrid	3	2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
13 Amand	0	−3	0	0	2	−2	1	0	0	0	0	−1	0	0	0	0	0	3
14 Hugh	3	0	0	−3	0	0	0	−2	0	0	0	1	0	0	2	0	−1	0
15 Boniface	0	3	−2	−1	0	0	1	0	0	0	0	2	−3	0	0	0	0	0
16 Albert	0	3	−1	−3	0	0	2	0	0	0	0	0	0	0	1	0	−2	0
17 Elias	0	1	2	−1	0	−3	0	−2	0	0	0	0	0	0	0	0	0	3
18 Simplicius	0	1	2	−1	0	0	0	−3	0	−2	0	0	0	0	0	0	3	0

be 'negative' errors whenever there are negative arcs within plus-sets. If n^k_{1m} is the numerical value of a negative arc from l to m within C_k, then the contribution of C_k to the negative error is $\Sigma_{1,m} n^k_{1m}$. Contributions to the 'positive' error come from positive arcs between plus-sets. Let C_r and C_s be distinct clusters. Additionally, let i be a vertex in C_r and j a vertex in C_s. We use p^{rs}_{ij} to denote the numerical value of a positive arc from i to j. The contribution of C_r and C_s to the positive error is $\Sigma_{i,j} p^{rs}_{ij}$. Weighting the positive and negative equally, we have:

$$P(C) = \sum_k \left(\sum_{1,m} n^k_{1m} \right) + \sum_{r,s} \left(\sum_{i,j} p^{rs}_{ij} \right) \tag{3}$$

and if positive and negative errors are not weighted equally, we use:

$$P(C) = \alpha \left[\sum_k \left(\sum_{1,m} n^k_{1m} \right) \right] + (1-\alpha) \left[\sum_{r,s} \left(\sum_{i,j} p^{rs}_{ij} \right) \right] \tag{4}$$

with α as defined previously.

If strict structural balance is the focus of attention, then we need to consider partitions into two clusters, consistent with Theorem 3. However, if we consider the kind of partitioning suggested by Davis (1967), then we can look at partitions with any number of clusters, consistent with Theorem 4. Table 5 shows the values of the optimized criterion function for partitions with up to five clusters.

The second row of Table 5 presents the measures of imbalance of 21.5, 16.0 and 12.5 for the successive time points. It is clear that the amount of imbalance in the structure declined steadily through time. However, it is interesting to note that the optimized criterion function is larger for partitions with two clusters than it is for partitions with three clusters. This suggests that the generalized form of

Table 5
Measures of imbalance through time

Number of clusters	Time Points		
	T_2	T_3	T_4
1	48.5	48.0	47.0
2	21.5	16.0	12.5
3	17.5	11.0	10.5
4	19.0	13.5	12.5
5	20.5	16.0	15.0

partitioning is more appropriate for this network. The measure of (generalized) imbalance also declines through time.

At T_2 there is only one partition with the minimized criterion function value of 17.5. One cluster is made up of John Bosco, Gregory, Mark, Winfred, Hugh, Boniface and Albert. This collection of actors is exactly the *Young Turks* identified by Sampson. The second cluster is made up of Peter, Bonaventure, Berthold, Victor, Ambrose, Romuald and Louis. With one exception, this is the *Loyal Opposition* identified by Sampson. Finally, the third cluster is made up of Basil, Elias, Simplicius and Amand. This group contains all of the *Outcasts*. The only difference between this partition and the one reported by Sampson is the location of Amand among the *Outcasts* rather than belonging to the *Loyal Opposition*. At Time T_3 there is again only one partition with the optimized criterion function value (of 11). This partition is exactly the partition obtained at T_2. For the final time period, there are two partitions having the minimum criterion function value of 10.5 and one of these is exactly the partition reported for the previous two time points [12]. It seems, then, that the partitioning of the actors is stable and that there has been a polarization process leading to increased 3-balance through time.

Table 6 shows the permuted affect matrices for the three points, where cluster members are adjacent [13]. It is immediately clear that there are very few negative links *within* the identified subgroups. *The Young Turks* have one at T_2 and two at T_4. *The Loyal Opposition* has one at T_2 and at T_3. Overwhelmingly, positive ties between subgroups contribute most to the error count. This suggests that negative ties within (polarized) subgroups are not tolerated while there is latitude for the formation of positive ties between such subgroups. Intuitively, this makes sense and is an hypothesis worth testing in other data.

Table 7 shows the ties inconsistent with generalized balance ($k = 3$) for each of the time points. Consistent with the decline of the imbalance measure shown in Table 5, the number of these ties also declines through time. At T_2, there were 20 arcs identified as contributing to the index of imbalance. Two of these were negative links within identified clusters as noted above, and 18 were positive links going between clusters. For T_3, there were only 14 ties identified as contributing to

[12] The other partition with a minimum value of the criterion function is considered subsequently.
[13] The null ties (0) are marked with · (as attention is focused on the non-null ties).

the generalized imbalance. Finally, at T_4 the number of links contributing to the generalized imbalance has dropped to 11. These are macro-level items. As was the case with the tiny four actor system of Fig. 4, identifying the discrepant (with

Table 6
Plus-set partitions (for k = 3) of the actors at each time point

Actor	id	Actors																	
		1	2	7	12	14	15	16	3	13	17	18	4	5	6	8	9	10	11
T_2																			
John Bosco	1	·	·	−1	·	1	·	·	2	·	·	·	·	3	−2	·	·	−3	·
Gregory	2	3	·	2	·	1	·	·	·	−3	−2	·	·	·	·	·	·	−1	·
Mark	7	·	2	·	·	·	·	3	·	·	·	·	−3	−1	−2	1	·	·	·
Winfrid	12	3	2	·	·	1	·	·	−1	·	·	·	−3	·	·	·	·	·	−2
Hugh	14	3	·	·	2	·	2	·	·	−3	−1	·	·	·	·	−2	·	·	1
Boniface	15	3	2	·	·	1	·	·	−2	−3	−1	−1	·	·	·	·	·	·	·
Albert	16	1	2	3	·	·	·	·	·	−1	−3	−2	·	·	·	·	·	·	·
Basil	3	2	3	·	·	·	·	·	·	·	1	·	−1	·	·	−3	−2	·	·
Amand	13	·	−3	1	−1	·	·	·	·	·	·	3	·	2	−2	·	·	·	·
Elias	17	·	·	·	·	·	·	·	3	2	·	1	−3	−2	·	·	·	·	−1
Simplicius	18	2	3	1	·	·	·	−1	·	·	·	·	−3	·	−2	·	·	·	·
Peter	4	·	·	−3	·	·	·	·	−2	·	·	−1	·	3	1	·	·	2	·
Bonaventure	5	·	·	·	·	·	·	·	·	1	·	·	3	·	·	·	·	·	2
Berthold	6	1	·	−3	−2	·	·	·	·	·	·	·	3	·	·	−1	2	·	·
Victor	8	3	2	·	·	−2	·	·	−3	·	−1	·	·	·	·	·	1	·	·
Ambrose	9	·	·	·	·	·	·	1	−3	·	−2	−1	·	2	·	3	·	·	·
Romuald	10	·	·	·	·	2	·	·	·	·	·	·	3	·	·	1	·	·	·
Louis	11	·	·	·	·	2	·	·	−1	−3	−2	·	·	3	·	1	·	·	·
T_3																			
John Bosco	1	·	2	·	·	·	·	·	·	·	·	·	−2	1	·	3	·	−3	−1
Gregory	2	3	·	1	2	·	1	·	·	−1	·	·	−3	·	·	−2	·	·	·
Mark	7	1	2	·	·	·	·	3	·	·	·	·	−3	·	−2	−1	·	·	·
Winfrid	12	3	1	·	·	2	·	·	−3	·	−1	·	·	·	−2	·	·	·	·
Hugh	14	3	1	·	1	·	2	·	·	·	−1	·	−3	·	·	·	·	−2	·
Boniface	15	2	3	·	·	1	·	·	−2	−3	−1	·	·	1	·	·	·	−1	·
Albert	16	·	2	3	1	·	·	·	−1	·	−2	·	−3	·	·	·	·	·	·
Basil	3	3	−1	·	·	·	·	·	·	·	1	2	·	·	−2	−3	·	·	·
Amand	13	·	−3	1	−1	·	·	·	·	·	·	3	·	2	−2	·	·	·	·
Elias	17	·	1	·	·	·	·	·	2	·	·	3	−2	·	−1	·	·	·	−3
Simplicius	18	·	1	·	·	·	·	·	·	2	3	·	−3	·	−2	·	·	·	−1
Peter	4	−2	−3	·	·	·	·	·	·	·	·	·	·	3	2	·	·	·	1
Bonaventure	5	2	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	1
Berthold	6	1	·	·	−2	·	·	·	−3	·	·	·	3	·	·	−1	2	·	·
Victor	8	−2	−3	·	·	·	·	·	−1	·	·	·	3	·	1	·	·	2	·
Ambrose	9	·	·	·	2	·	·	·	−3	·	−2	−1	·	1	·	3	·	·	·
Romuald	10	·	·	·	·	·	·	·	·	2	·	·	3	1	·	·	·	·	·
Louis	11	−1	−3	·	·	·	·	1	−2	·	·	·	2	3	·	·	·	·	·

Table 6 (continued)

Actor	id	Actors																	
		1	2	7	12	14	15	16	3	13	17	18	4	5	6	8	9	10	11
T_4																			
John Bosco	1	·	−2	−3	1	2	·	·	3	·	·	·	·	·	·	·	·	−1	·
Gregory	2	3	·	1	2	·	·	·	·	−1	·	·	−3	·	·	−2	·	·	·
Mark	7	·	3	·	1	·	·	2	·	·	·	·	−3	·	−2	−1	·	·	·
Winfrid	12	3	2	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
Hugh	14	3	·	·	1	·	2	·	·	·	−1	·	−3	·	·	−2	·	·	·
Boniface	15	·	3	1	2	·	·	·	−2	−3	·	·	−1	·	·	·	·	·	·
Albert	16	·	3	2	·	·	1	·	−1	·	−2	·	−3	·	·	·	·	·	·
Basil	3	3	−2	·	·	·	·	−1	·	2	1	2	−3	·	−2	·	·	·	·
Amand	13	·	−3	1	−1	·	·	·	·	·	·	3	·	2	−2	·	·	·	·
Elias	17	·	1	·	·	·	·	·	2	·	·	3	−1	·	−3	−2	·	·	·
Simplicius	18	·	1	·	·	·	·	·	2	·	3	·	−1	·	·	−3	·	−2	·
Peter	4	−2	−3	·	·	−1	·	·	·	·	·	·	·	3	1	·	·	·	2
Bonaventure	5	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	1	·	2
Berthold	6	·	−1	−2	·	·	·	·	−3	·	−2	·	3	1	·	·	2	·	·
Victor	8	·	−3	·	·	−1	·	·	−2	·	·	·	3	·	2	·	1	·	·
Ambrose	9	·	·	·	2	·	·	·	−3	·	−2	−1	·	1	·	3	·	·	·
Romuald	10	·	·	·	·	·	·	·	·	2	·	·	3	1	·	·	1	·	·
Louis	11	−1	−3	·	·	1	·	·	−2	·	·	·	·	2	·	3	·	·	·

k-balance) ties does not allow us to predict with certainty which ties will actually change through time.

6. Summary

The use of the optimization procedure for the affect data from Sampson (1968) has yielded useful information. First, it has identified a meaningful partition, into three groups of actors, that has the lowest values of the criterion function for generalized balance at each time point. Second, it has provided a measure of imbalance for each of the time points, and third, it has identified those links that contribute to the imbalance found in the partitioned structures. These outputs provide a straightforward way of testing one of the key theoretical tenets of structural balance and have suggested an additional hypothesis concerning the distribution of the ties that are inconsistent with balance.

In addition, the clusters that are identified with this method are extremely close to those delineated by Sampson and found also with blockmodelling procedures. The only difference is the location of Amand among the Outcasts in this analysis. There is evidence in Sampson's narrative that this is the appropriate location for Amand. Sampson (1968, p. 354) describes a secret ballot with four candidates. Surprisingly, Basil, given his location among the *Outcasts*, is one of the candidates. Of interest, here, is the fact that Amand nominated Basil. From subsequent

Table 7
Ties identified as inconsistent with 3-balance

Value	Ties identified
T_2	
(−1)	John Bosco → Mark
(+2)	John Bosco → Basil
(+3)	John Bosco → Bonaventure
(+1)	Mark → Victor
(+1)	Hugh → Louis
(+2)	Basil → John Bosco
(+3)	Basil → Gregory
(+1)	Amand → Mark
(+2)	Amand → Bonaventure
(+2)	Simplicius → John Bosco
(+3)	Simplicius → Gregory
(+1)	Simplicius → Mark
(+1)	Bonaventure → Amand
(+1)	Berthold → John Bosco
(−1)	Berthold → Victor
(+3)	Victor → John Bosco
(+2)	Victor → Gregory
(+1)	Ambrose → Albert
(+2)	Romuald → Hugh
(+2)	Louis → Hugh
T_3	
(+1)	John Bosco → Bonaventure
(+3)	John Bosco → Victor
(+1)	Boniface → Bonaventure
(+3)	Basil → John Bosco
(+1)	Amand → Mark
(+2)	Amand → Bonaventure
(+1)	Elias → Gregory
(+1)	Simplicius → Gregory
(+2)	Bonaventure → John Bosco
(+1)	Berthold → John Bosco
(−1)	Berthold → Victor
(+2)	Ambrose → Winfrid
(+2)	Romuald → Amand
(+1)	Louis → Albert
T_4	
(−2)	John Bosco → Gregory
(−3)	John Bosco → Mark
(+3)	John Bosco → Basil
(+3)	Basil → John Bosco
(+1)	Amand → Mark
(+2)	Amand → Bonaventure
(+1)	Elias → Gregory
(+1)	Simplicius → Gregory
(+2)	Ambrose → Winfrid
(+1)	Romuald → Amand
(+1)	Louis → Hugh

interviews, Sampson was able to reconstruct the vote and the only actor voting for Basil was Amand. At all three time points, there are no negative links within the *Outcast* cluster identified here. Further, the number of positive ties among them grows at each time point. We note that White et al. (1976, p. 750) report the same partition of the 18 actors obtained from blockmodelling all of the relational ties for T_4.

The analyses presented in Tables 4–7 use the valued data. As balance theory was formulated for binary signed graphs, it is useful to repeat the analysis with binary ties only. Recoding each 2 and 3 in Table 4 to 1 gives the binary matrices. When these are analyzed — using Eq. (2) — the partitions are identical. The pattern of the $+1$ and -1 elements is the same as the positive and negative elements in Table 6. All that changes is the numerical value of the criterion function [14] for T_2, T_3 and T_4 for three clusters are 10, 7 and 5.5, respectively. In general, the analyses of valued and binary matrices will not be identical. In this case, however, the results are robust.

At T_4 there was a second partition — displayed in Table 8 — with the minimum value of the criterion function. The *Loyal Opposition* remains intact but two members of the *Young Turks* are reclassified with the *Outcasts*.
Given the consistency of the other partition through the three time periods, it seems reasonable to pursue the partition in Table 6. However, both do fit equally well with a criterion function value of 10.5. The partition in Table 8 has 13 ties identified as inconsistent with generalized balance, instead of the 11 identified in Table 6 and 7. The analysis of the binary data yields only the partition in Table 6, T_4.

7. Discussion

The classic paper of Cartwright and Harary (1956) contained the result that if the graph/network is 2-balanced, the macro-structure of the graph had two mutually antagonistic subgroups having internal solidarity. Davis (1967) extended the formulation to one where there can be more than two such mutually antagonistic subgroups. As the macro-structure is described in terms of a partition, it is fruitful to use a partitioning approach to structural and generalized balance. The basic structure theorems were used to specify a criterion function that was then optimized via a local optimization procedure. This procedure yields the partition(s) closest to a k-balanced state, one with a coherent measure of imbalance of the overall network and an identification of the minimal set of ties that are inconsistent with a balanced state. All of which is useful information concerning the macro-structure. These methods were used on the Sampson (1968) monastery data with convincing results. Along the way, the local optimization procedure provided

[14] Instead of adding the 2s and 3s, only 1s are added for the binary matrices.

Table 8
An alternative narrative for the affect ties at T_4 (a) Alternative partition with optimized criterion function

Actor	id	1	3	13	14	17	18	2	7	12	15	16	4	5	6	8	9	10	11
John Bosco	1	·	3	·	2	·	·	−2	−3	1	·	·	·	·	·	·	·	−1	·
Basil	3	3	·	2	·	1	2	−2	·	·	·	−1	−3	·	−2	·	·	·	·
Amand	13	·	·	·	·	·	3	−3	1	−1	·	·	·	2	−2	·	·	·	·
Hugh	14	3	·	·	·	−1	·	·	·	1	2	·	−3	·	·	−2	·	·	·
Elias	17	·	2	·	·	·	3	1	·	·	·	·	−1	·	−3	−2	·	·	·
Simplicius	18	·	2	·	·	3	·	1	·	·	·	·	−1	·	·	−3	·	−2	·
Gregory	2	3	·	−1	·	·	·	·	1	2	·	·	−3	·	·	−2	·	·	·
Mark	7	·	·	·	·	·	·	3	·	1	·	2	−3	·	−2	−1	·	·	·
Winfrid	12	3	·	·	·	·	·	2	1	·	·	·	·	·	·	·	·	·	·
Boniface	15	·	−2	−3	·	·	·	3	1	2	·	·	−1	·	·	·	·	·	·
Albert	16	·	−1	·	·	−2	·	3	2	·	1	·	−3	·	·	·	·	·	·
Peter	4	−2	·	·	−1	·	·	−3	·	·	·	·	·	3	1	·	·	·	2
Bonaventure	5	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	1	·	2
Berthold	6	·	−3	·	·	−2	·	−1	−2	·	·	·	3	1	·	·	2	·	·
Victor	8	·	−2	·	−1	·	·	−3	·	·	·	·	3	·	2	·	1	·	·
Ambrose	9	·	−3	·	·	−2	−1	·	·	2	·	·	·	1	·	3	·	·	·
Romuald	10	·	·	2	·	·	·	·	·	·	·	·	3	1	·	·	1	·	·
Louis	11	−1	−2	·	1	·	·	−3	·	·	·	·	·	2	·	3	·	·	·

(b) Ties Identified as Inconsistent with 3-balance

Value	Tie
(+1)	John Bosco → Winfrid
(+1)	Amand → Mark
(+2)	Amand → Bonaventure
(−1)	Hugh → Elias
(+1)	Hugh → Winfrid
(+2)	Hugh → Boniface
(+1)	Elias → Gregory
(+1)	Simplicius → Gregory
(+3)	Gregory → John Bosco
(+3)	Winfrid → John Bosco
(+2)	Ambrose → Winfrid
(+2)	Romuald → Amand
(+1)	Louis → Hugh

a confirming [15] test of a basic tenet of k-balance theory: signed social structure tends towards k-balance. For these data, the generalized structure theorem (with $k = 3$) was more useful than the original theorem for structural balance.

[15] The Sampson data used here were collected via a survey tapping recalled data for sociometric items. This may have a built-in bias towards balance, weakening such a test. However, the data analytic methods are not threatened by this.

The identification of ties inconsistent with generalized balance at a time point is not powerful enough to be a predictive tool for locating which ties will change subsequently. This is not unreasonable. If the balance mechanism does work in triadic configurations, then, as each actor is involved in many triads, changes towards balance in one triad may move other triads into imbalance. If there is a force field, in the sense of Lewin (1951), then it is inappropriate to expect that these micro-processes can be modeled with the macro-level tools employed here. We do emphasize that the study of these micro-processes, and the way they 'cumulate' to generate macro-structural properties remain critical areas of study.

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