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Partitioning Signed Two-Mode Networks

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Structural balance theory forms the foundation for a generalized blockmodel method useful for delineating the structure of signed social one-mode networks for social actors (for example, people or nations). Heider's unit formation relation was dropped. We re-examine structural balance by formulating Heider's unit formation relations as signed two-mode data. Just as generalized blockmodeling has been extended to analyze two-mode unsigned data, we extend it to analyze signed two-mode network data and provide a formalization of the extension. The blockmodel structure for signed two-mode networks has positive and negative blocks, defined in terms of different partitions of rows and columns. These signed blocks can be located anywhere in the block model. We provide a motivating example and then use the new blockmodel type to delineate the voting patterns of the Supreme Court justices for all of their nonunanimous decisions for the 2006–07 term. Interpretations are presented together with a statement of further problems meriting attention for partitioning signed two-mode data.

Keywords: generalized blockmodeling, signed networks, signed two-mode networks, structural balance

1. INTRODUCTION

Structural balance theory has its primary origins in the work of Heider (1946, 1958) who considered some of the potential dynamics in signed *pox*-triples for two social actors $\{p, o\}$ and a third social object $\{x\}$ and, in principle, *poq*-triples for three actors $\{p, o, q\}$ where q is

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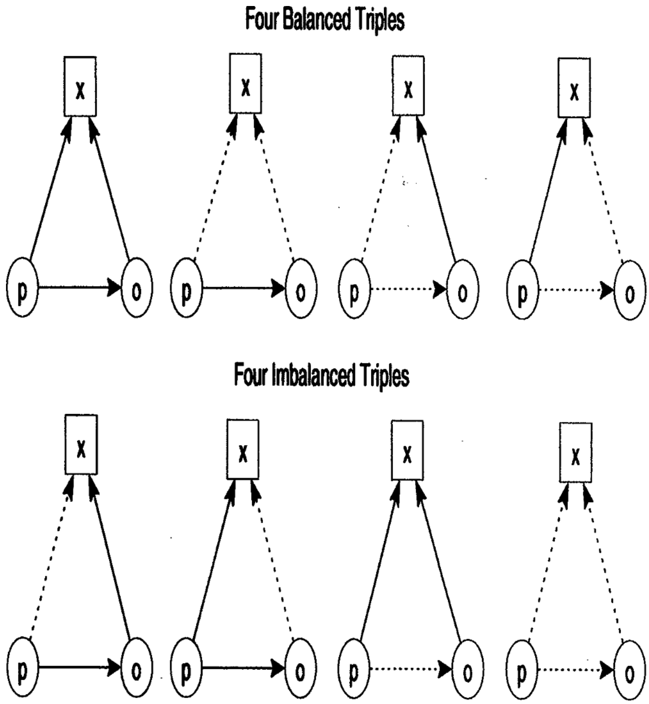

FIGURE 1 Balanced and imbalanced *pox* triples.

the third actor. Examples of *pox*-triples are shown in Figure 1 where solid lines represent positive ties and dashed lines represent negative ties. For our purposes here, the primary kind of social object, x, is a belief or a decision. In *pox*-triples the relations between $\{p, o\}$ and $\{x\}$ are, in Heider's terms, unit formation relations[1]: the ties of unit relations associate actors with social objects.

Regarding beliefs, people can accept them (positive unit formation ties) or reject them (negative unit formation ties) or remain neutral (null unit formation ties). An example featuring decisions has justices on courts, including the U.S. Supreme Court, who make rulings with individual justices agreeing with specific decisions (positive links) or dissenting from them (negative links). Because *pox*-triples contain

[1]In contrast, for *poq*-triples there is only one kind of social entity—people—and the ties are social relations (and perceptions of them). The asymmetry in terms of types—actors and social objects—was discarded as part of the generalization of Heider's ideas in order to focus attention on social relations.

two kinds of social entities, we have represented the two types of objects as ellipses and boxes in Figure 1 as a reminder of their intrinsic difference. Suppose x represents a belief that is important to both p and o. Heider argued that balanced triples are stable and provide no strain for individuals having them. When p has a positive tie to o, they agree about x, and when the tie from p to o is negative they disagree regarding x. This holds for all triples in the top row of Figure 1. On the other hand, the imbalanced triples in the bottom row of Figure 1 are sources of strain and tension for p according to Heider. In these triples, when p likes o they disagree about x and when p dislikes o they agree about x. These triples will be unstable.[2] A similar logic holds for *poq*-triples of individuals. There is some ambiguity about the all negative triple in the bottom row of Figure 1, especially for *poq* triples. In essence, for *poq*-triples, Davis (1967) suggested considering them to be balanced and stable.

Cartwright and Harary (1956) formalized Heider's approach and proved a remarkable theorem regarding the overall structure of a network of signed ties. This caught fire as an idea: balanced signed one-mode networks have a characteristic structure of breaking into mutually hostile subgroups, later called plus-sets by Davis (1967), where all of the positive ties occur with plus-sets while all of the negative ties occur between members in different plus-sets.[3] This led to a stream of research to determine these plus-sets, describe the overall network structure and measure the amount of imbalance in the network. For temporal signed one-mode networks attention was focused on whether the amount of imbalance declined over time given the dynamic nature of Heider's theory. Doreian and Mrvar (1996) proposed a partitioning method based on the characteristic structure depicted in the formal theorems of Cartwright and Harary (1956) and Davis (1967). In turn, their approach was absorbed into generalized blockmodeling (Doreian et al., 2005).

Even though the formalization provided by Cartwright and Harary provided the foundations for a productive line of research delineating the group structure of signed networks, it came at a cost. Their treatment of network structures was done in terms of signed relations.

[2]Of course, it depends on the importance of x. Depending on the circumstances, it would seem that agreeing or not about the merits of U2 as a band is less consequential than disagreeing or not over the morality of an unprovoked invasion of another country.

[3]For Cartwright and Harary, a network is balanced if it has *two* plus-sets, one of which can be empty, with this characteristic sign pattern. For Davis, a graph was clusterable into two or more plus-set with this sign pattern. We prefer to use the term k-balance where k is the number of plus-sets.

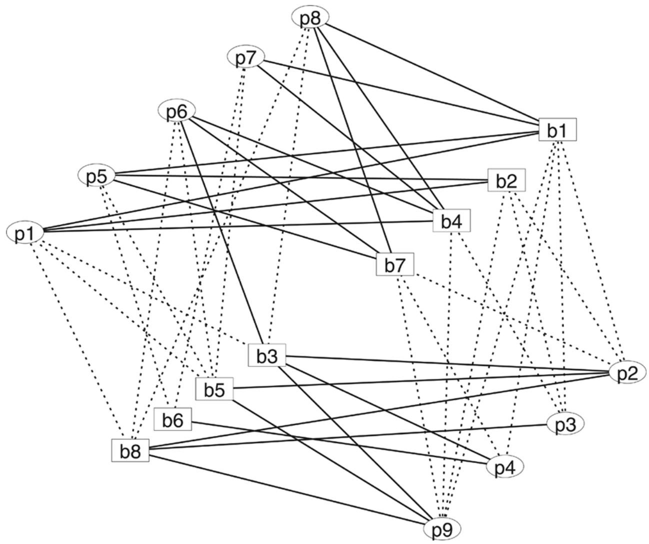

FIGURE 2 An artificial two-mode of actors and beliefs.

This meant that the distinction between social relations between two people and unit formation relations between people and social objects, such as beliefs, was discarded (Doreian, 2004). In practice, this meant that attention was confined to observed (or reported or inferred) signed social relations. Also, much of Heider's formulation was presented in terms of cognitive dynamics inside people's minds. This also was discarded in the graph theoretic formulation. The simulations of Hummon and Doreian (2003) that featured both social relations and perceptions of relations, provided some additional insight into the complexities of Heiderian balance processes. It is clear that there is much to be gained by returning to reconsider the unit formation relations. We do this by formulating the idea of signed two-mode networks as a natural representation of them.

Consider the simple example in Figure 2 of nine people {p1 through p9} and their acceptance or rejection of a set of beliefs {b1 through b8} drawn as a bipartite graph. Circles represent people and squares beliefs. Table 1 gives the same information in a formatted matrix grouped according to the common acceptance or rejection of beliefs, allowing for some individuals to have no opinion on some beliefs (which is represented by the 0 elements). In this constructed example, there are two subgroups of people and two subsets of beliefs. Five people {p1, p5, p6, p7, p8} accept {b1, b2, b4, b7} and reject {b3, b5,

TABLE 1 Partition of Two-Mode Data for Actors and Beliefs

	b1	b2	b7	b4	b5	b6	b3	b8
p1	1	1	0	1	−1	0	−1	−1
p5	1	1	1	0	−1	−1	0	0
p6	0	0	1	1	−1	0	*1*	−1
p7	1	0	0	1	−1	−1	0	0
p8	1	0	1	1	0	0	−1	−1
p2	−1	−1	−1	0	1	0	1	1
p3	−1	−1	0	−1	0	0	0	1
p4	−1	0	−1	0	0	1	1	0
p9	−1	−1	−1	−1	1	0	1	1

b6, b8}. Four people {p2, p3, p4, p9} have the reverse pattern. (The one exception is p6 who accepts b3 and this link is in italics in Table 1.) The partition shown in Table 1 represents the sort of partition we need to get from a partitioning algorithm for such signed two-mode data. Figure 2 shows the ties as edges rather than arcs because we treat the unit formation relation as one where social actors are associated with beliefs. Our concern is to cluster both the social actors and the social objects (e.g. beliefs or decisions) assuming that some social objects are shared by some actors while other social objects are shared by other actors.

2. FORMALIZATION OF BLOCKMODELING SIGNED TWO-MODE DATA

Borgatti and Everett (1992) proposed the idea of blockmodeling multiway, multimode networks. Within the rubric of generalized blockmodeling, Doreian et al. (2004, 2005) extended the idea of blockmodeling from one-mode networks to two-mode networks. This extension was picked up and pushed further by Brusco and Steinley (2006, 2007). Applying this idea to signed two-mode networks follows the same logic. Structural balance theory, as based on the Cartwright and Harary formulation and mobilized within generalized blockmodeling, has been applied exclusively to one-mode data. Following Doreian et al. (2005), a directed (binary) *signed one-mode network* is an ordered pair, $(\mathbf{G}, \sigma)$, where:

1. $\mathbf{G} = (\mathcal{V}, \mathcal{A})$ is a digraph, without loops, having a set of vertices (nodes), $\mathcal{V}$, and a set of arcs, $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$, and

2. $\sigma : \mathscr{A} \rightarrow \{p, n\}$ is a sign function. The arcs with the sign p are positive while the arcs with the sign n are negative. An alternative notation, consistent with most diagrams of signed social one-mode networks, is $\sigma : \mathscr{A} \rightarrow \{+1, -1\}$.

If the signed network had edges instead of arcs, the definition becomes: An undirected (binary) *signed one-mode network* is an ordered pair, $(\mathbf{G}, \sigma)$, where:

1. $\mathbf{G} = (\mathscr{V}, \mathscr{E})$ is a graph, without loops, having a set of vertices (nodes), $\mathscr{V}$, and a set of edges,[4] $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and
2. $\sigma : \mathscr{E} \rightarrow \{p, n\}$ is a sign function. The edges with the sign p are positive while the edges with the sign n are negative. An alternative notation, consistent with most diagrams of signed social one-mode networks, is $\sigma : \mathscr{E} \rightarrow \{+1, -1\}$.

In the following, we use edges rather than arcs in the definition of a signed two-mode network. Anticipating the analysis to follow for the Supreme Court, with justices and the cases they hear, the justices form one set of social entities while the decisions (made regarding cases) form the other. All justices are linked to a decision by their votes[5] as unit formation ties. We conceptualize such a link as an *edge* even though it is possible to think of a justice as supporting or dissenting from a decision. When they vote against a decision in the minority (i.e., dissent) they are linked by negative edges to the case. The set of all justices, the cases and their votes (as edges) form an undirected signed two-mode network structure.[6]

One part of the basic theorem underlying structural balance partitions for signed one-mode data is due to Cartwright and Harary (1956). See also Harary, Norman, and Cartwright (1965) for the complete theorem.

Theorem 1. *A signed network* $(\mathbf{G}, \sigma)$ *is balanced if and only if every closed semiwalk is positive.*

Following Roberts (1976), a signed network $(\mathbf{G}, \sigma)$ is *partitionable* if and only if the set of vertices, $\mathscr{V}$, can be partitioned into clusters so

[4]We represent edges as a pair of reciprocated arcs as a way of capturing the notion of unordered pairs (two element sets). Formally, balance theoretic partitioning can proceed in terms of arcs, edges or both.

[5]When justices recuse themselves from particular cases, there are null unit formation ties.

[6]The same formalization holds, in principle, when arcs are used rather than edges.

that every positive arc (or edge) joins vertices of the same cluster and every negative arc (or edge) joins vertices of different clusters.

With this reformulation, the above theorem generalizes to (Davis, 1967).

Theorem 2. *A signed network* $(\mathbf{G}, \sigma)$ *is exactly partitionable into two or more plus-sets if and only if it contains no closed semiwalk with a single negative line.*

As stated, these theorems concern the partitioning of a *single* set of vertices. For two-mode signed networks there are *two* sets of vertices, each of which is partitioned. In a blockmodeling approach, vertices and relational ties are partitioned simultaneously. The clusters that partition the vertices are equivalence classes, called *positions*. For pairs of positions, the set of ties between the vertices in them is called a *block*. When partitions of one-mode data have k positions, there are k^2 blocks. For partitions of two-mode data, where there are k_1 positions for one set of vertices and k_2 positions for the other set, there are $(k_1 \times k_2)$ blocks. In setting up the generalized blockmodeling approach, Doreian et al. (2005) took concepts of structural equivalence (Lorrain and White, 1971) and regular equivalence (White and Reitz, 1983) and translated them into permitted block types, given a particular equivalence, and fitting a blockmodel to data by using the block types in a direct approach. Part of their generalization was to define equivalences in terms of sets of new blocks types (Doreian et al., 2005, Chapter 6). We pursue the same idea here by seeking partitions of the set of social actors, denoted by $\mathcal{U}$, and the set of social objects, denoted by $\mathcal{V}$, and considering the blocks they define in signed two-mode data as being positive or negative. Ideally, the positive blocks contain only positive or null ties while the negative blocks contain only negative or null ties.

For two-mode signed data, the balance formulation must be changed to accommodate two sets of social entities, each represented by a set of vertices rather than one. We use the following notation: let $\mathcal{U} = \{u_1, u_2, \ldots, u_{n_1}\}$ represent the vertices in the first set and $\mathcal{V} = \{v_1, v_2, \ldots, v_{n_2}\}$ the vertices in the second set. A binary undirected *signed two-mode network* is an ordered pair, $(\mathbf{G}, \sigma)$, where:

1. $\mathbf{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$ is a bipartite graph having two non-empty sets of vertices, $\mathcal{U}$ and $\mathcal{V}$, and a set of edges, $\mathcal{E} \subseteq \mathcal{U} \times \mathcal{V}$, and
2. $\sigma : \mathcal{E} \to \{p, n\}$ is a sign function. The edges with the sign p are positive while the edges with the sign n are negative. An alternative notation for this is $\sigma : \mathcal{E} \to \{+1, -1\}$.

We denote the sizes of $\mathcal{U}$ and $\mathcal{V}$, respectively, by n_1 and n_2. Let k_1 be the number of clusters for social entities in $\mathcal{U}$ and k_2 the number of clusters of social entities in $\mathcal{V}$. Clearly, $1 \leq k_1 \leq n_1$ and $1 \leq k_2 \leq n_2$. We term the partition with k_1 and k_2 clusters a (k_1, k_2)-partition of the signed two-mode data. This means that there is a partition of $\mathcal{U}$ into k_1 clusters and a partition if $\mathcal{V}$ into k_2 clusters. As the full term is cumbersome, we use also a briefer version, *signed two-clustering*. For a specific partition (when the values of k_1 and k_2 are clear) we refer to this as a "two-clustering." Again, k_1 and k_2 refer the number of clusters of, respectively, $\mathcal{U}$ and $\mathcal{V}$.

In blockmodeling a signed two-mode network $\mathbf{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$ we are trying to identify a two-clustering $\mathbf{C} = (\mathbf{C}_1, \mathbf{C}_2)$ where $\mathbf{C}_1$ is a partition of $\mathcal{U}$ and $\mathbf{C}_2$ is a partition of $\mathcal{V}$, such that they induce positive and negative blocks. We denote the set of all feasible[7] two-clusterings by Φ.

Following Doreian et al. (2004, 2005), the two-mode generalized blockmodeling problem is set up as an optimization problem $(\Phi, P, \min)$: Determine the set of two-clusterings $\mathbf{C}^\star = (\mathbf{C}_1^\star, \mathbf{C}_2^\star) \in \Phi$ for which $P(\mathbf{C}^\star) = \min_{(\mathbf{C} \in \Phi)} P(\mathbf{C})$ where Φ is the set of feasible two-clusterings and P is the criterion function. A clustering (for one-mode data) or two-clustering (for two-mode data) is feasible if it satisfies all of the requirements imposed by a specified block structure of the blockmodel being fitted to the data.

Under the Cartwright and Harary (1956) formulation, the vertices of a balanced one-mode network can be partitioned into exactly two clusters (positions), and under the Davis (1967) generalization the vertices can be partitioned into two or more clusters. We use k to denote the number of clusters in a partition of a signed network. A k-balanced network is one that is partitioned into k clusters where the positive ties are between vertices in the same cluster and the negative ties are between vertices in different clusters. (see footnote 3). This is called an ideal k-balance partition for a one-mode signed network. For a two-mode signed network, an ideal (k_1, k_2)-partition is one where the positive blocks contain no negative ties and the negative blocks contain no positive ties. Put differently, in an ideal (k_1, k_2)-partition there are no blocks with both positive and negative ties. For signed networks (regardless of whether they are one-mode or two-mode) the criterion function is defined in terms of positive and negative blocks via a count of elements that are *not* consistent with an ideal k-balance partition (for one-mode data) or an ideal (k_1, k_2)-partition

[7]See Doreian et al. (2005, pp. 185–186), for the meaning of this term for one-mode data. The idea extends natural to two-mode partitions and consists of all partitions corresponding to the set of all possible exact partitions given a type of equivalence.

(for two-mode data). These inconsistencies take the form of negative ties within positive blocks and positive ties within negative blocks.[8] Let $\mathcal{N}$ be the total number of negative ties within positive blocks and let $\mathcal{P}$ be the total number of positive ties within negative blocks. A general criterion function is: $P(\mathbf{C}) = \alpha\mathcal{N} + (1-\alpha)\mathcal{P}$ where $0 \le \alpha \le 1$. With $\alpha = 0.5$, the two inconsistencies are equally weighted. For $0 \le \alpha < 0.5$, positive inconsistencies are viewed as more important and for $0.5 < \alpha \le 1$, the negative inconsistencies are seen as more important. For the data in the artificial example, as shown in Table 1, $k_1 = k_2 = 2$. Also, $P(\mathbf{C}) = 0.5$ due to one tie (the positive edge linking p6 and b3) known to be inconsistent with perfect consistency of beliefs for the two subgroups and $\alpha = 0.5$.

For partitioning one-mode signed data, the behavior of the fitting procedure for blockmodels is given by Theorem 3 (Doreian et al., 2005, Chapter 10). With k the number of plus-sets in a partition ($1 \le k \le n$), partitions with k and $k+1$ plus-sets are said to be adjacent and

Theorem 3. *For any signed one-mode network,* $(\mathbf{G}, \sigma)$, *there will be a unique lowest value of the criterion function. This value will occur for partitions with a single number* (k) *of plus-sets or for adjacent partitions.*

For each value of k, there will be an optimal partition. Given Theorem 3 for some value(s) of k there will be an overall optimal value of $P(\mathbf{C})$. If there is one partition with this optimal value, it is unique. However, it is possible that there are multiple equally well fitting optimal partitions.

The behavior of $P(\mathbf{C})$ in terms of k_1 and k_2 for the set of optimal (k_1, k_2)-partitions does not behave according to Theorem 3. Instead, its behavior for partitioning signed two-mode data is given by Theorem 4.

Theorem 4. *Given a signed two-mode graph* $\mathbf{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$ *and the set of optimal partitions for signed* (k_1, k_2)*-partitions, with* $1 \le k_1 \le n_1$ *and* $1 \le k_2 \le n_2$*, the optimal values of* $P(\mathbf{C} = (\mathbf{C}_1, \mathbf{C}_2))$ *decline monotonically with* k_1 *for each value of* k_2 *and monotonically with* k_2 *for each value of* k_1.

[8]Viewing inconsistencies in this fashion is an application of the line indices of imbalance proposed by Harary et al. (1965, pp. 348–352). An alternative approach using proportions of balanced cycles does not address the partitioning problem. Further, counting cycles is computationally too complex (Hummon and Fararo, 1995) to be applied here.

Proof. Consider an optimal (k_1, k_2)-partition with k_1 row clusters and k_2 column clusters, $\mathbf{C}$, and denote the best[9] $P(\mathbf{C} = \mathbf{C}_{1(k_1)}, \mathbf{C}_{2(k_2)})$ by c_{k_1,k_2}. Consider next, a best partition with $k_1 + 1$ row clusters that has been obtained from the (k_1, k_2)-partition by splitting (partitioning into two subsets) a row cluster of that partition, keeping k_2 fixed. This split induces splits in the blocks for the split cluster for each column. Denote the new value of the criterion function by $P(\mathbf{C} = \mathbf{C}_{1(k_1+1)}, \mathbf{C}_{2(k_2)}) = c_{k_1+1,k_2}$. Consider a block split into two subblocks. If this was a negative block (had more -1s than $+1$s) split into two negative subblocks the same $+1$s will remain inconsistencies and $c_{k_1+1,k_2} = c_{k_1,k_2}$. However, if a subblock of a negative block is now a positive block then there must be more $+1$s than -1s and the criterion function will drop below c_{k_1,k_2}. Hence, $c_{k_1+1,k_2} < c_{k_1,k_2}$ and $P(\mathbf{C} = \mathbf{C}_{1(k_1+1)}, \mathbf{C}_{2(k_2)}) < P(\mathbf{C} = \mathbf{C}_{1(k_1)}, \mathbf{C}_{2(k_2)})$. Taking both types of change following the split of a cluster, $P(\mathbf{C} = \mathbf{C}_{1(k_1+1)}, \mathbf{C}_{2(k_2)}) \leq P(\mathbf{C} = \mathbf{C}_{1(k_1)}, \mathbf{C}_{2(k_2)})$. A similar argument holds for a positive block split into two subblocks with the role of $+1$s and -1s reversed. Also, a similar argument holds for splitting a column for a best (k_1, k_2)-partition while keeping k_1 fixed. This splitting process leads to best values of the criterion function that are not larger than the criterion function for the clusters before the split.

There can be instances where a best $(k_1 + 1, k_2)$-partition exists that is not nested within a best (k_1, k_2)-partition. (Given two partitions with different numbers of clusters, the fine-grained partition is nested within the coarse grained partition if every one of its clusters is contained within a cluster of the coarse grained partition.) Consider moving from a (k_1, k_2)-partition to $(k_1 + 1, k_2)$-partition. Suppose there is a best $(k_1 + 1, k_2)$-partition not nested inside a best (k_1, k_2)-partition and, for this partition, $P(\mathbf{C} = \mathbf{C}_{1(k_1+1)}, \mathbf{C}_{2(k_2)}) = d_{k_1+1,k_2}$. By the above argument, a best (k_1, k_2)-partition can be split to create a partition where $P(\mathbf{C} = \mathbf{C}_{1(k_1+1)}, \mathbf{C}_{2(k_2)}) = c_{k_1+1,k_2}$. If $d_{k_1+1,k_2} > c_{k_1+1,k_2}$ then the original partition is not 'best'. It follows that $d_{k_1+1,k_2} \leq c_{k_1+1,k_2}$. A similar argument holds for movement to a $(k_1, k_2 + 1)$-partition from a (k_1, k_2)-partition. Empirically, there may be instances where there are an equal number of positive and negative ties in a block. This can be handled in the same fashion. □

This result is a slight extension of the result for one-mode signed networks in Doreian and Mrvar (2009), and the behavior of the

[9]This is the optimal value of the criterion function for these specific values of k_1 and k_2. We use "best" in this sense throughout the proof. For given values of k_1 and k_2, the value of $P(\mathbf{C} = \mathbf{C}_{1(k_1)}, \mathbf{C}_{2(k_2)})$ is optimal but need not be globally optimal.

criterion function over the number of positions (clusters) is similar to the corresponding criterion function for structural equivalence (Doreian et al., 2005, 187). For structural balance partitions of one-mode signed data, there is minimum value of the criterion function for some value(s) of k where $1 < k < n$ (see Doreian et al., 2005, Theorem 10.6). This unique value of the criterion function defines a clear stopping rule for the partitioning. In contrast, the behavior of $P(\mathbf{C})$ for partitioning signed two-mode network data given by Theorem 4 implies that there is no clear stopping rule, a contrast with partitioning signed one-mode network data in terms of structural balance. Of course, a partition where $k_1 = n_1$ and $k_2 = n_2$ will have a criterion function of 0 but this partition has no practical merit.

3. AN EXTENDED EMPIRICAL EXAMPLE

The example we use is taken from the 74 decisions[10] data made by the Supreme Court for the 2006–07 term for which the set of justices is {Stevens, Ginsburg, Souter, Breyer, Kennedy, Alito, Roberts, Scalia, Thomas}. The voting behavior of the justices was obtained from the Supreme Court Web site (http://www.supremecourtus.gov/opinions/06slipopinion.html) and reading the opinions with regard to content and establishing who agreed with which decision and who dissented from each decision when they were not unanimous. Opinions filed by the justices were examined to see if, while not concurring with the majority opinion, they still agreed with the decision while providing a different rationale to the one provided by the majority. At face value, only strict legal concerns enter into a justice's vote and, while there may be differences in opinions and interpretations of legal issues and precedent, this leads to an expectation that there would be no systematic differences in the ways that justices vote across a set of cases that come before the Supreme Court in a single term. The intense political battles that occur when Congress considers candidates nominated by a sitting president (deParle, 2005; Edsall, 2005) suggest that such an expectation is naive. If politics and ideology are relevant in the nomination and deliberation over installing someone on the Supreme Court, then it is reasonable to expect systematic patterns of voting behavior once justices are on this court. These patterns, when they exist, can be delineated by using the partitioning approach outlined in Section 2.

[10]Doreian et al. (2004, 2005) used only the "important" decisions for their partitions of the Supreme Court in the years they considered. The implications of using the full set of decisions are discussed below.

Our primary concern here is to use the 2006–07 Supreme Court data described above to demonstrate the partitioning of signed two-mode networks with less emphasis on interpreting all of the patterns that we delineate in terms of the politics of the court and the legal issues coming before it. However, some interpretation is provided. It is worth noting at the outset that there are 28 cases that were decided on a unanimous basis, albeit with individual justices recusing themselves for some of them. For these cases, it is reasonable to argue that the legal issues involved are clear enough with regard to the constitutionality of the enacted laws, or lower court decisions, featured in them.[11] Our interest here centers on the nonunanimous cases where there are, by definition, differences among the justices serious enough for them to disagree publicly in their support for, or dissent from, the decisions taken. There were 46 cases featuring such disagreements. The essential question is whether or not there are systematic patterns among the justices and also the cases. Answers come in the form of partitioning the signed two-mode data for these cases. Pajek (Batagelj and Mrvar, 1998) was used for the signed two-mode blockmodeling reported here. This specific partitioning was included in Pajek, Version 1.20, in June 2007. (See also de Nooy et al., 2005, for an extended discussion of using Pajek.)

Hitherto, blockmodeling of network data, regardless of whether they are two-mode or one-mode networks, results in partitions applying to the *whole* network. For one-mode data, the rows and columns are partitioned in the same way for the whole network. For two-mode data, even if the partitions of the rows and columns are necessarily different, they apply to the whole two-mode data array. Given that the nonunanimous decisions of the Supreme Court can range from 5–4 decisions to 8–1 decisions, it makes little sense to seek partitions of the whole array for the justices and their decisions. Put differently, to claim there is a *single* partition of the justices and a *single* partition of the decisions seems a severe over-simplification. We consider a series of subsets of cases that are organized on a principled basis to allow for differences to be delineated in the ways that sets of cases are organized when different numbers of justices are in the majority. We start by considering 5–4 cases.

[11]Even so, different justices can have different rationales for supporting a unanimous decision.

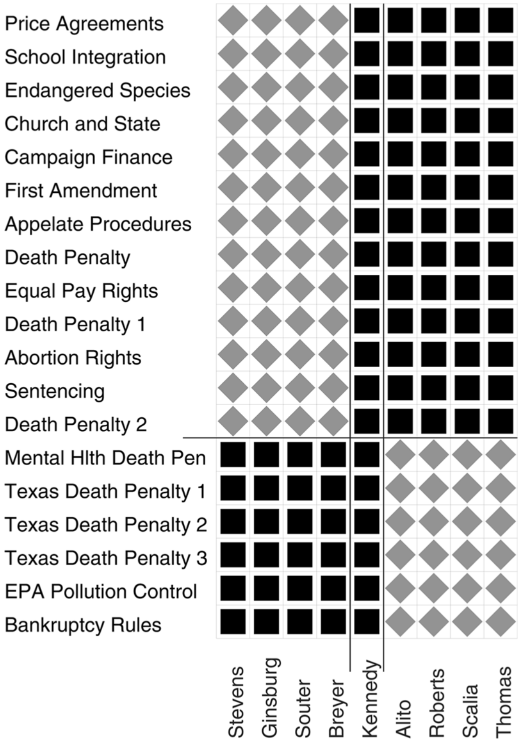

FIGURE 3 Partition of signed two-mode data for ideological cases.

3.1. Clear Ideological Cases

Figure 3 shows the two-mode partition of 19 cases with a 5-4 majority where both the so-called "conservative" and the so-called "liberal" wings of the court voted together.[12] Justice Kennedy agreed with one side or the other depending on the issues considered. Using the partitioning method for signed two-mode data, this (2, 3)-partition is unique with $P(\mathbf{C}) = 0$. Positive votes are marked with squares and the negative votes by diamonds. On these cases the conservative wing prevailed on 13 with the liberal wing in the majority for only six, three of which are reversals of particular death sentences from Texas.

Some of the cases attracted more media attention than others. One of the "big" cases included the School Integration case.

[12]Henceforth we will use the label conservative in place of so-called conservative and liberal in place of so-called liberal.

The conservative justices {Alito, Roberts, Scalia, Thomas} showed great hostility toward affirmative action and were in the majority when they struck down redistricting plans for two school districts (whose cases had been joined together) that considered race in only a secondary fashion. The issues surrounding abortion rights are highly contentious and, in this case, the conservatives prevailed by declaring the so-called partial birth abortion method as unconstitutional. Another notable decision came in the Equal Pay Rights cases where an earlier lower court judgment in favor of a woman who suffered long-term wage discrimination by her employer was struck down. Conservatives on the court generally were in favor of imposing and upholding death sentences and making it easier to impose the death penalty at sentencing in capital offences. The pattern shown here merits the description of the 2006–07 Supreme Court as having liberal and conservative wings—with Justice Kennedy located between them—that were divided by these ideological decisions. The partition of the cases can be interpreted in this light. The block pattern for this partition is:

Negative	Positive	Positive
Positive	Positive	Negative

The middle column of positive blocks in the signed blockmodel image contain only the positive votes (voting in the majority) of Kennedy on all of these issues.

Figure 4 shows the bipartite graph for these ideological cases. The cases and justices are grouped according to the partition shown in Figure 3. The cases won by the conservatives are shown in the left column with those won by the liberal wing on the right column. The justices are shown in the middle column where the liberal wing justices are at the top, the conservative wing justices at the bottom and Justice Kennedy is in the middle. As will be clear from the following empirical results, the meaning of the term “center of the court” or the “middle of the court” is inherently ambiguous and depends on the cases involved. For the decisions featured in Figure 4 it is reasonable to claim that Justice Kennedy was the center of the court.

3.2. Pairs of Justices at the Extremes

Next, we consider a set of 10 cases where two pairs of justices appeared together in the minority. The center of the court for *these* cases has Justice Kennedy joined by two justices from the liberal

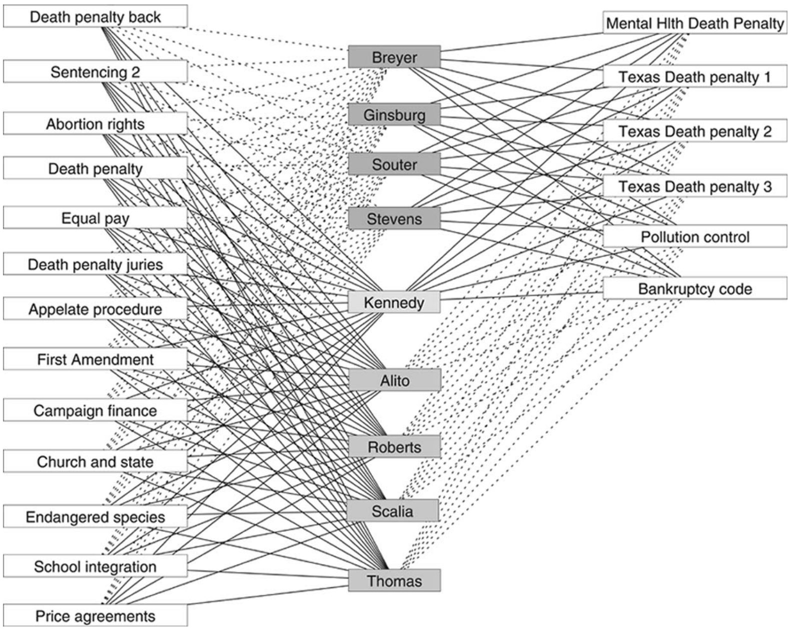

FIGURE 4 Bipartite graph of partitions for ideological cases.

wing and two justices from the conservative wing of the court. Two members {Stevens, Ginsburg} of the liberal wing are in their own cluster and {Scalia, Thomas}, from the conservative wing, appear in their own cluster. For these cases, this center prevails with five members who were usually in the majority. The exceptions are Justice Alito who was in the minority for the Death Penalty Appeal case (which is the only contribution to the nonzero $P(\mathbf{C}) = 0.5$ for this unique partition) and Justice Breyer who took no part in the False Claims Act case (which does not contribute to the $P(\mathbf{C})$). The cases are partitioned into three sets. For the two cases in the top cluster of Figure 5, both pairs {Stevens, Ginsburg} and {Scalia, Thomas} are in dissent. The next cluster of five cases has only {Scalia, Thomas} in dissent with {Stevens, Ginsburg} joining the center. Finally, the three cases in the bottom cluster of cases has {Stevens, Ginsburg} alone in dissent. One partial interpretation is that Justices Stevens and Ginsburg are more liberal than the other two justices of the liberal wing while Justices Scalia and Thomas are more conservative than the other members of the conservative wing. It is an incomplete interpretation because the two extreme pairs joined *together* in dissent

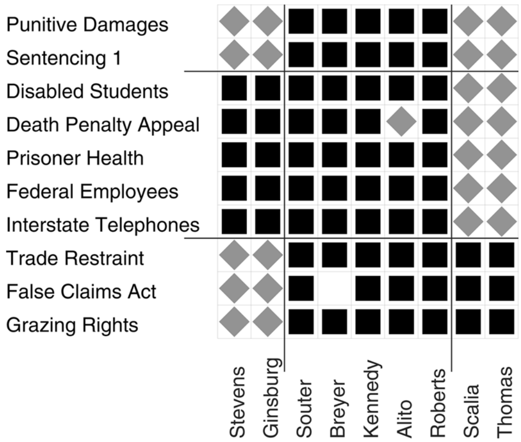

FIGURE 5 Partition of signed two-mode data for pairs of justices at extremes.

for two of these cases. The block pattern for these cases is:

Negative	Positive	Negative
Positive	Positive	Negative
Negative	Positive	Positive

Figure 6 shows the bipartite graph version of this two-mode signed network. The five justices always in the majority are in the center of the middle column. The pair {Stevens, Ginsburg} from the liberal wing, who dissent together, is at the top of the middle column while the pair {Scalia, Thomas} for the conservative wing is shown at the bottom of this column. The set of cases where Justices Scalia and Thomas dissent are shown in the column on the right. The three cases where Justices Stevens and Ginsburg dissent are on the bottom of the left hand column. The two cases where both pairs of justices dissent are at the top of this column. For these decisions, the center of the court is comprised of {Souter, Breyer, Kennedy, Alito, Roberts}.

3.3. Single Justices at the Extremes

For the 8–1 decisions, we have justices dissenting alone from a decision that the rest of their colleagues support. In one sense, this does point to a form of being extreme characterized by a willingness to stand alone publicly and be seen as being at odds with the remaining

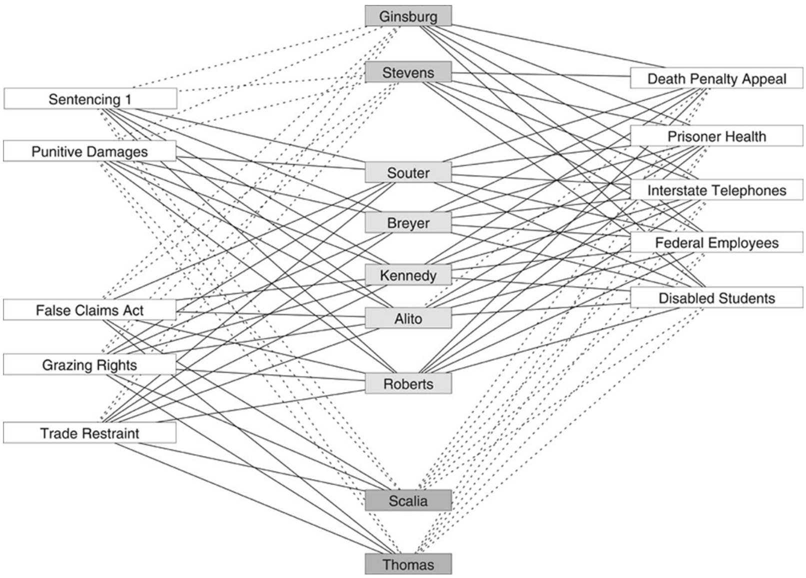

FIGURE 6 Bipartite graph of partitions with pairs of justices at extremes.

justices. Figure 7 shows an optimal (2, 3)-partition for a set of six cases where either Justice Stevens or Justice Thomas stands alone. The center of the court is made up of the remaining seven justices. The block structure of this partition is:

Positive	Positive	Negative
Negative	Positive	Positive

These results suggest that it is tenable to think there are variations in the extent to which justices belong to one of the two wings of the court. The cases with pairs of justices who were found in dissent, allowing for the qualification that there were two cases where they dissented together, then {Stevens, Ginsburg} and {Scalia, Thomas} comprised the extremes of the liberal and conservative wings. The cases with singletons in the extremes, sharpens this slightly and suggests that Justice Stevens is at the extreme of the liberal wing while Justice Thomas is the most extreme in the conservative wing for the 2006–07 term.

In the bipartite two-mode network shown in Figure 8, the left hand column has the cases where Justice Stevens dissented when Justice Thomas joined the majority, while the right hand column has the

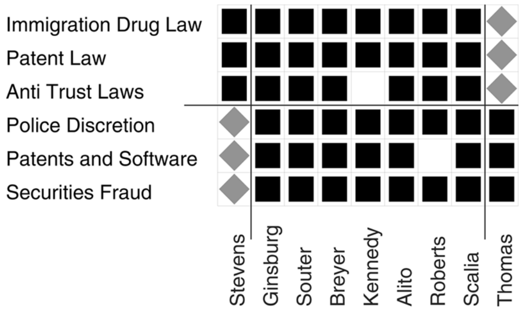

FIGURE 7 Partition of signed two-mode data for single justices at extremes.

reverse when Justice Stevens joined the majority and Justice Thomas dissented. The center column has the seven justices always in the majority in the middle with Justice Stevens at the top of the column and Justice Thomas at the bottom.

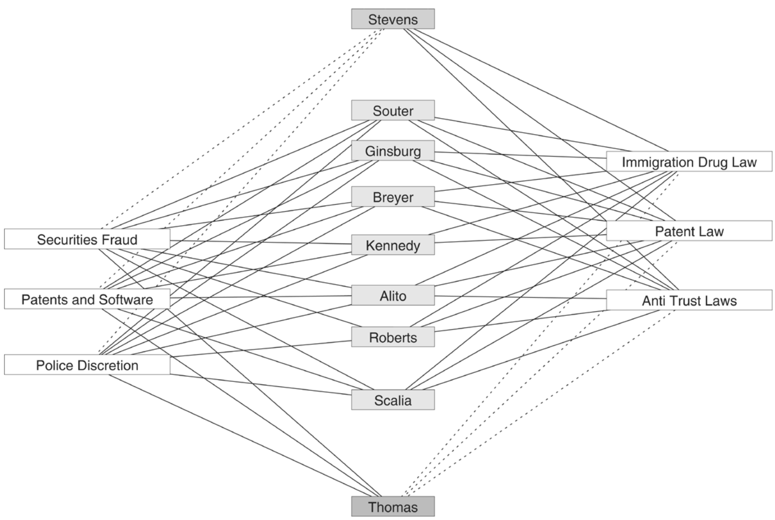

FIGURE 8 Bipartite graph of partitions with single justices at extremes.

3.4. One Solid Wing and One Fractured Wing

Thus far, we started with a subset of cases which, when partitioned, revealed a Supreme Court with two solid wings whose members voted

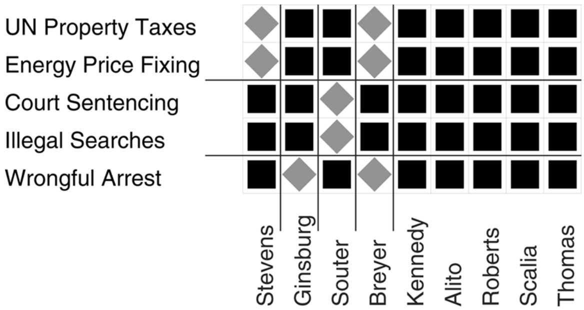

FIGURE 9 Partition with a solid wing and a fractured wing.

together as a block. Justice Kennedy, in essence, became the deciding justice when he joined one wing or the other. We then looked at cases where each of the wings lost members to the center leaving two pairs, one from each wing, dissenting on another subset of decisions. That was taken one step further with another subset of cases having single justices, one from each wing, voting in dissent. The cases considered in this section form yet another pattern as shown in Figure 9. This is a more fine grained partition with three clusters of cases and five clusters of justices. On the right hand side of this figure, the conservative wing together with Justice Kennedy form the core of the majority. The members of the liberal wing join them on occasion but their voting pattern is such that they are each singletons in a cluster. The UN Property Tax and Energy Price Fixing cases have Justices Ginsburg and Souter joining the majority leaving Justices Stevens and Breyer in dissent. The Court Sentencing and Illegal Search cases see Justice Souter alone in dissent while the Wrongful Arrest case has Justices Ginsburg and Breyer in dissent. The conservative wing is solid for these cases while the liberal wing is totally fractured. For this small set of cases, it would appear that the label "liberal" is not accurate given that, selectively, its members join the conservative wing of the court in deciding particular cases. For these decisions, it is reasonable to see the conservative wing and Justice Kennedy as the center of the court.

3.5. Cases with No Clear Pattern

The remaining set of five nonunanimous decisions reveals no coherent pattern with regard to the voting of the justices. There is an exact partition shown in Figure 10 with $P(\mathbf{C}) = 0$ but with 5 clusters of

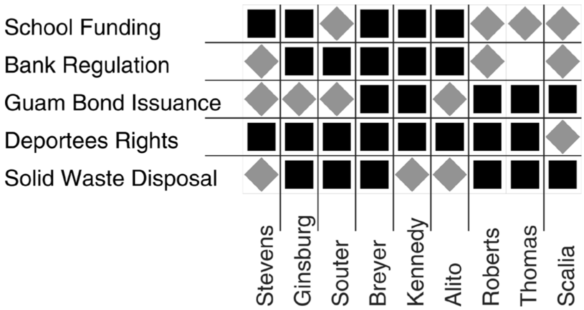

FIGURE 10 Five cases having no clear partition structure.

the decisions and eight clusters of justices it is hardly a reduction. It preferable to recognize that they display no pattern. The only "reduction" is that Justices Roberts and Thomas are clustered together[13] while the remaining justices are singletons in their own clusters. Justice Breyer, alone, is in the majority for all of these cases. There is no simplifying reduction from partitioning these cases and partitioning reveals an absence of patterns in the voting behavior of the justices. For these cases there is no meaning to the terms center nor wings of the court. Such a pattern, albeit for a small minority of the cases, conforms to the naive expectation that there are no systematic differences between the justices when deciding cases.

3.6. A Nontraditional Two-Mode Blockmodel

The matrix arrays for the partitioned two-mode matrices from Subsections 1 through 4 are joined together into a single blockmodel in Figure 11. The ideological cases forms the foundation at the bottom with its partition of the court into two wings and Justice Kennedy between them. The second panel from the bottom has the pairs of dissenting justices on either side of a middle make up of Justice Kennedy with colleagues drawn from the two wings. The next panel up has Justices Stevens and Thomas as distinct singletons as solo dissenting justices. At the top of Figure 11 is the panel with one solid wing and one fractured wing. The dividing lines between the justices in their clusters extend into the spaces between panels. Note that the display is possible because there is no permutation of the

[13]While Justice Thomas recused himself from the case, the null cell does not contribute to $P(\mathbf{C})$.

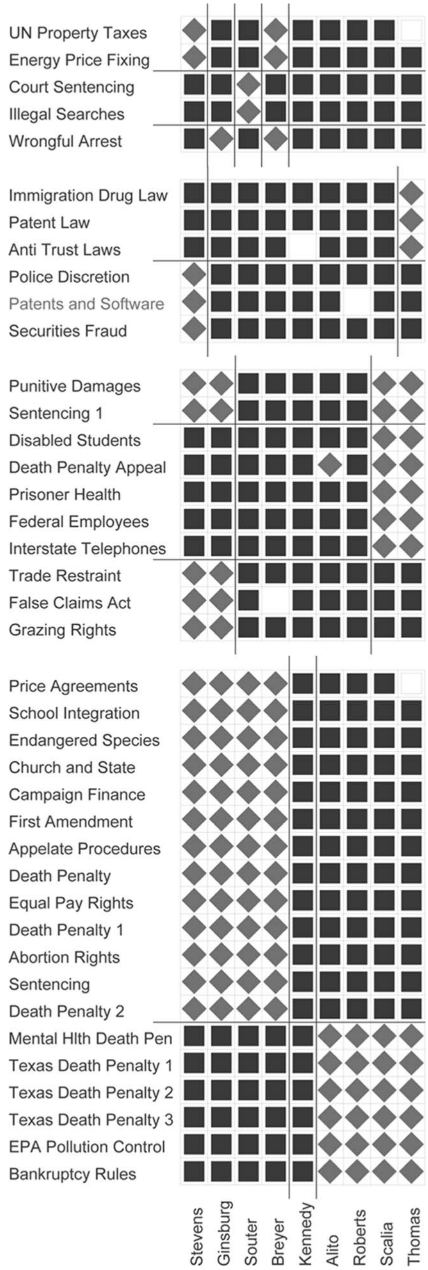

FIGURE 11 A nontraditional blockmodel with four blockmodels.

justices even though the clusters of them differ in each panel. The count of inconsistencies for this nontraditional blockmodel is easily constructed from each of the component subblockmodels. This total[14] 1 and is the sum of 0 (from Fig. 3), 1 (from Fig. 5), 0 (from Fig. 7) and 0 (from Fig. 9). While very low, this (constructed) total results from the construction of the separate sets of cases and the exclusion of those cases with no coherent pattern. For the US Supreme Court with nonunanimous decisions (ignoring the null unit formation ties) ranging from 5–4 through 8–1, the selection of subsets of cases is straightforward. For more general signed two-mode data sets the decision process will be less straightforward.

4. DISCUSSION

This paper has presented four new ideas: (a) conceptualizing the *pox*-triples of the Heiderian structural balance theory as two-mode data with edges rather than arcs which leads naturally to the notion of signed two-mode data, (b) extending generalized two-mode blockmodeling to such signed two-mode data, (c) providing a new type of blockmodel together with a method for fitting it, and (d) relaxing the idea that the clusterings of a blockmodel apply across all of the data. The result is an extension of generalized blockmodeling to include signed two-mode data where the blockmodel takes the form of having positive and negative blocks appearing anywhere in it. Rows and columns are partitioned separately, but at the same time, to define the blocks. We have also provided a theorem and its proof establishing the basic behavior of the criterion function, $P(\mathbf{C} = (\mathbf{C}_1, \mathbf{C}_2))$, in relation to k_1 and k_2, the number of row and column clusters, respectively. This approach was demonstrated using the data on the nonunanimous decisions made by the Supreme Court for the 2006–07 term.

Even though the Supreme Court data were used primarily to demonstrate the idea of signed two-mode networks and how to partition them, the results have interest in their own right. First, the range in the margin of the nonunanimous decisions is from 5–4 to 8–1 making it less than useful to seek a single partition of the decisions, and of the justices, and expect them to hold across the whole two-mode data matrix. The behavior of the justices is too complicated for such a strategy. The structure of their collective behavior is more varied than the simple image of a conservative wing and a liberal

[14]This is the number of inconsistencies and the value of the criterion function, $P(\mathbf{C}) = \alpha\mathcal{N} + (1-\alpha)\mathcal{P}$ is 0.5 with $\alpha = 0.5$.

wing. Certainly, this characterization—with Justice Kennedy placed between the two wings—applies for 19 of 46 nonunamimous cases. It may be a basic characterization around which there are nuances and variations. There may be a set of cases that achieve prominence because enough people care about them and mobilize to support or oppose particular decisions. Affirmative action with regard to race and gender, abortion rights and the death penalty appear to be such issues and the 5–4 decisions on such ideological cases do provide ammunition for thinking in terms of a fundamental divide. There are also the 28 unanimous decisions to temper this conclusion. (Were they included in the blockmodeling effect, the result would be uninteresting with just a single cluster of decisions and justices.) The other sets of cases considered here also revealed different patterns and variations, all of which would have been lost if a single blockmodel for all cases had been fitted. At a minimum, the conventional characterization of the U.S. Supreme Court as having conservative and liberal wings seems too simple and lacks relevant nuances. If only the "important" cases are considered, it appears that these are deemed important because constituencies have been mobilized around them as part of an ideological dispute. The partition shown in Figure 3 represents this division. But not all cases have this form and tend to be ignored when only a sample of cases is considered. No doubt, further interpretations could be made in terms of the detailed nature of the cases[15]—but this is not our concern here. This does suggest an intriguing future avenue of work. While we have focused solely upon partitioning of the signed two-mode data, there are two related one-mode networks that merit attention. One concerns Justice-to Justice relations that could be constructed in terms of social ties (a difficult task) and influence relations. The other concerns the actual cases where their content constructs both similarities and dis-similarities.

There remain problems to solve or issues to address. One is the collection of rich enough data for larger signed two-mode structures. In addition to the issue of the size of the data set, the Supreme Court data are rather dense with the support and dissent being virtual complements of each other. Ignoring the dissent altogether, as Doreian et al. (2004, 2005) did when they used structural equivalence partitioning of two-mode data, would most likely lead to comparable results. Given that the methods for partitioning two-mode data

[15]It is a future topic of research to consider the details of the cases to discern if there are coherent ideological commonalities leading to systematic differences in voting behavior.

proposed by Brusco and Steinley (2006, 2007) are for unsigned two-mode data, it would appear that their approach is not directly applicable. However, for the rather simple structure of the Supreme Court, using their approach with only the majority votes is likely to produce similar results. When the number of null elements increase, the performance of structural equivalence methods and those based on signed relations, as proposed here, will diverge in general. Part of a data collection effort could include a systematic examination of signed belief systems that are more complex than the Supreme Court data.

A second problem is the creation of a sound stopping rule. Partitioning one-mode signed network data in terms of structural balance has a clear stopping rule with a well defined minimum value for the criterion function which follows from Theorem 3. The method proposed here lacks this and establishing one seems useful. A third issue is to take seriously one feature that can emerge when structural equivalence is employed: the presence of null blocks. Finally, just as Doreian et al. (2004, 2005) suggested the application of two-mode methods to partition one-mode data, together with one example of doing so, we anticipate examining one-mode signed data with this method defined for signed two-mode data.

A fourth issue, involves coupling of the basic signed two-mode data with two (or more) one-mode networks[16] defined for the two sets of social units. Making the analyses more dynamic would be useful with a view to making the analyses predictive. This would involve delving into the Justice-to-Justice network(s) and the content links between cases. The predictive aspect could be accommodated within years by examining the sequence of decisions that are announced (although this might not map cleanly into the order they were actually decided) and across years. This points to a need to couple signed two-mode data to one-mode data sets capturing intranodal properties as a general strategy and doing this in a dynamic fashion. This will be a nontrivial matter because these dynamics are subtle. That the value of the criterion function was so low for the nontraditional blockmodel shown in Figure 11 suggests that Heiderian principles are at work, albeit not in the stark form of Figure 3 where an ideological divide, one demanding consistency on both sides, is evident. However, they are not the only "force at work" because the justices also consider legal issues concerning (their interpretation of) the U.S. Constitution when making decisions. This leads to the cases where some justices are

[16]These are *not* the two one-mode networks that can be constructed from this two-mode network.

willing to stand alone in their dissents from decisions made by their colleagues. Heiderian consistency principles may be more evident in data sets featuring of beliefs and actors not inhabiting an arena so public as the U.S. Supreme Court.

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