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Actor network utilities and network evolution

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This paper is dedicated to the memory of Norman P. Hummon.

Abstract

Considerable attention has been devoted to the evolution of social networks through time. Some has been descriptive, some has sought fundamental principles for studying network evolution and some has been analytical. The work described in this paper represents all three approaches and is cast within the framework of rational choice theory. One foundation is found in the work of Jackson and Wolinsky [JW] [Jackson, M.O., Wolinsky, A., 1996. A strategic model of social and economic networks, Journal of Economic Theory, 71, 44–74] who consider actors having some calculus for the costs of maintaining social ties and the benefits received by virtue of being located in a network of ties. By parameterizing the costs and benefits, they derived equilibrium structures under specified combinations of parameters when ties are formed by rational actors. The second foundation is found in the simulations of Hummon [Hummon, N.P., 2000. Utility and Dynamic Social Networks, Social Networks, 22, 221–249.] that were based on the JW work. Hummon found that there were equilibrium structures that were not anticipated in the formal analyses of JW. To resolve this discrepancy, using networks with a fixed set of vertices and using the JW framework, this paper explores the transitions between networks on the lattices of all graphs with a fixed number of vertices through the addition and deletion of ties. While these transitions can be described, a close examination of them reveals the equilibrium structures anticipated by JW, the equilibrium structures located by the Hummon simulations, plus some other equilibria. Modified theorems for the equilibria forms are presented together with a generalization of the equilibrium concept.

Keywords: Social network; Network evolution; Actor

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Social network analysts once focused primarily on describing network structure within a static framework. This changed dramatically when they focussed explicitly on network processes in social networks. While descriptions of network changes constituted an important first step, there is now considerable activity focused on social processes and the evolution of social networks. Doreian and Stokman (1997: p. 3) took the view that "network processes are series of events that create, sustain and dissolve social structures." This theme was picked up in Doreian (2002) with emphasis on series of events as potential generators of social structure which is an approach consistent with Abell's (1987) approach to narratives of event sequences generating outcomes and structures. Examples of other work with a focus on generating social structural forms, in a stronger analytic form, include (Chase, 1982; Fararo et al., 1994).

As Monge and Contractor (1987) note, social theorists have had a long standing in self-interest as a motivation for human action. It is assumed that individuals guided by self-interest make social choices in order to benefit from those choices. This line of thought evolved, in part, to a concern with social capital (e.g. Coleman, 1990; Lin et al., 2001) where access to the resources of others over network ties becomes a useful resource for human actors. Here, we restrict attention to the utility of the ties themselves (without considering the resources to which these ties provide access). Stokman and Doreian (1997: p. 235) note that for some network studies, "the underlying process for network change is assumed to be located in the network structure" within which social actors are located. They also argued that networks have an instrumental character for network members because these members have structural goals and some goals are achieved through network choices. The current paper takes these two insights as a point of departure for a sustained look at social actors making network choices that are conditioned by the structure of the network within which they are located. Self-interest is assumed to govern network choices of rational actors.

Jackson and Wolinsky (1996), hereafter JW, provide an elegant argument for, and demonstration of, actors having some calculus for the costs of maintaining social ties and the benefits received by virtue of being located in a network of ties. By parameterizing the benefits and costs, JW showed how differing regimes of these parameters led to different network structures. Their primary interest was in the nature of the equilibrium distribution of ties and shapes of the networks at equilibrium. Hummon (2000) used an agent-based simulation approach to study the behavior of actors having the benefits, costs and resultant utilities for the network ties as specified by JW. Hummon's concern was focused on the generation of ties and the nature of network evolution through time.[1] His simulated actors created and dissolved network ties and thereby generated sequences of network structures. Much of the time, the network sequences did create the equilibrium forms described by JW. However, Hummon identified other equilibrium structures *not* anticipated by JW.

This appeared to call into question some aspects of the basic JW result and it is reasonable to examine the conditions under which other and perhaps sub-optimal structures are created – and to examine what prevents movement from them to optimal structures. Examining

[1] As noted by Hummon, this was not JW's concern. However, there is considerable merit in examining the trajectories of network forms (Doreian, 2002).

pathways for the evolution of structural forms, as located in a set of *potential* forms is a way of doing this. By exploring transitions between pairs of structures it is possible to establish the conditions under which structures not identified by JW occur and are stable. An additional assumption made here is that rational actors are satisficing in the sense of Simon (1976) and the notion of 'bounded rationality'. If actors do not have the time or knowledge to examine all alternatives exhaustively before making choices, the timing and sequencing of those choices become critical and their utility functions need not be optimized.

The set of all edge graphs for n vertices forms a lattice which identifies the transitions between graphs when lines are added (or deleted) one at a time. The graph G_p is contained in the graph G_q if it is a subgraph of G_q. It is an immediate lower neighbor if $G_p = G_q \setminus l_g$ where l_g is an edge in G_q. Similarly, if $G_q = G_p \bigcup l_g$, then G_q is an immediate upper neighbor of G_p. Of course, every pair of graphs in the inclusion lattice has a greatest lower bound and a least upper bound. Strictly, each G_p represents an equivalence class of isomorphic graphs.

1. Benefits, costs and utilities from network ties

Let δ_{ij} and γ_{ij} denote, respectively, the benefit for an actor, i, of the tie $(i \leftrightarrow j)$, henceforth denoted by (ij), and the cost of maintaining that tie for i. Actor j bears a similar cost, γ_{ji}, for the maintenance of this tie. From JW, if w_{ij} represents the value of actor j for actor i and t_{ij} the geodesic distance of j from i, the utility of a network, G, for i can be written as:

$$u_i(G) = w_{ii} + \Sigma_{ij\epsilon G}(\delta_{ij}w_{ij} - \gamma_{ij}) + \Sigma_{t_{ij}>1}(\delta_{ij}^{t_{ij}} w_{ij})$$

The first term, w_{ii}, captures the notion of self-worth. The second term is the total net benefit (or cost) for the *direct* ties involving i and the third term is the net benefit obtained from indirect paths between i and all other actors. However, in this third term, only geodesic paths count. From this equation, it is clear also that when there are multiple geodesics between i and j, there is only the term $\delta_{ij}^{t_{ij}} w_{ij}$, regardless[2] of the number of geodesics between i and j.

JW specified that the 'self' utilities, w_{ii}, be uniform over all i, say $w_{ii} = w, \forall i$. JW assumed that the total cost of maintaining a tie is the same for all ties. Accordingly, we let $\gamma_{ij} = \gamma$, $\forall i, j$, denote this cost and let $\delta_{ij} = \delta, \forall (ij) \in G$ be the uniform benefit that actors derive from each direct network tie involving them. The equation for actor utilities

[2] While it seems reasonable to allow all geodesics to have value, and so contribute to the utility of i, the equation as it stands is appropriate. Suppose there were r_{ij} geodesics between i and j, the last term in the equation for the actor utilities could be modified to $\Sigma_{t_{ij}>1}(r_{ij}\delta_{ij}^{t_{ij}} w_{ij})$ and used in place of the third term of the stated equation. Consider, for example, two geodesics $(ik_1)(k_1 j)$ and $(ik_2)(k_2 j)$ so $r_{ij} = 2$ with each geodesic contributing. However, it is necessary to examine the notion of weighting these geodesics. The example with two geodesics can be contrasted with the situation where there is only one geodesic $(ik)(kj)$. The two geodesics through k_1 and k_2 each appear less valuable than the single geodesic through k. It seems reasonable that they are equally important and are weighted by (1/2). Similarly, when there are r_{ij} geodesics between i and j, each should be weighted by $1/r_{ij}$. The inclusion of this weighting into the modified third term, leads to the equation as written.

becomes:

$$u_i(G) = w + \Sigma_{ij\epsilon G}(\delta w_{ij} - \gamma) + \Sigma_{t_{ij}>1}(\delta^{t_{ij}} w_{ij})$$

Further, if w_{ij} is uniformly 1, then the equation for the utility of a network to an actor, i, becomes

$$u_i(G) = w + \Sigma_{ij\epsilon G}(\delta - \gamma) + \Sigma_{t_{ij}>1}(\delta^{t_{ij}})$$

Because attention will center on comparisons of utilities for an actor in different network structures, we consider differences in utilities, for which w will always be subtracted out. The utilities listed throughout the tables omit the w term for notational clarity in these tables. The total utility from the network as a whole is given by

$$U_T = U_T(G) = \Sigma_i u_{i\epsilon G}(G).$$

Each such summation will have nw corresponding to the first term in the utility equation. For networks of a given size, n, these nw terms are not relevant for comparisons of utilities as they are subtracted out in these comparisons and are omitted from the tables giving total utilities (for networks of a given size).

In general, a value can be assigned to a graph and this particular specification of total utility is simply an example of assigning such a value. Letting G^n denote the set of all edge graphs for n points and $v(G)$ a function assigning values to graphs, a graph, G is strongly efficient if $v(G) \geq v(G'), \forall G' \epsilon G^n$. JW state and prove the following theorem.

Theorem 1. *The unique strongly efficient edge graph is*:

1. *the complete graph if* $\gamma < \delta - \delta^2$;
2. *a star on all n vertices if* $\delta - \delta^2 < \gamma < \delta + ((n-2)/2)\delta^2$;
3. *the empty graph on n vertices if* $\delta + ((n-2)/2)\delta^2 < \gamma$.

There appear to be some problems with this result and formulation. First, while $0 < \delta < 1$ is a reasonable range for benefits, the upper limit for γ increases with the group size and therefore without bound for the star structures. Second, Hummon (2000) identified other equilibrium structures beyond those listed in Theorem 1. We focus primarily on these two problems here. Third, for large n (or even moderate n), the structures specified in Theorem 1 are nonsensical because null, star and complete networks are seldom observed empirically. Additionally, it is not clear that each additional tie has the same utility for i as all ties formed before the additional tie. If it is the case that human actors can support a limited number of ties (through, for example, constraints imposed by time), there ought to be a declining marginal utility for additional ties. We examine these extra problems after considering the lattices of graphs for $3 \leq n \leq 5$. Underlying these analyses is a concern with the *sequences of decisions* made by social actors as they form and dissolve social ties and the possible transitions between graphs.

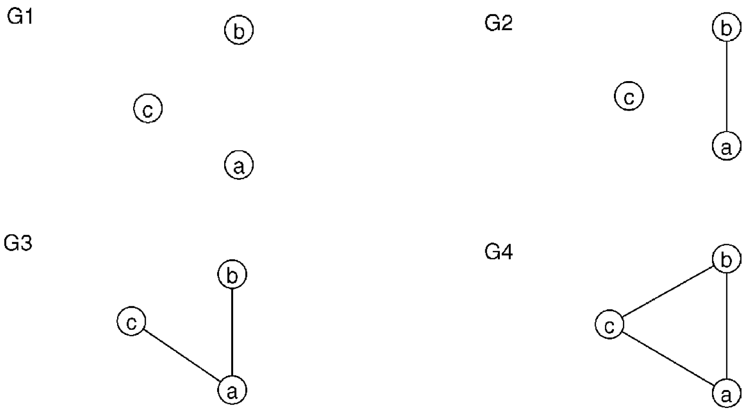

Fig. 1. All edge graphs with three vertices.

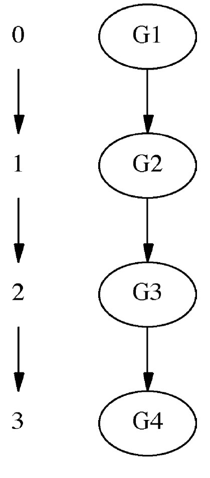

Fig. 2. Lattice of edge graphs with three vertices.

2. Graphs with three vertices

The four graphs (as representatives of isomorphism classes of graphs) with three vertices are shown in Fig. 1. The potential evolutionary sequence (as a lattice) with single line additions is given in Fig. 2. The numbers on the left (0 through 4) are the number of edges in a graph.

The utilities for all three actors in G_1 is zero as there are no ties. The graph G_2 is reached from G_1 with the addition of a single tie, between two actors, say a and b . The utility[3]

[3] Excluding the w term which is irrelevant for comparisons.

Table 1
Actor utilities for all graphs with three vertices

Graph	Actors		
	a	*b*	*c*
G_1	0	0	0
G_2	$(\delta - \gamma)$	$(\delta - \gamma)$	0
G_3	$2(\delta - \gamma)$	$(\delta - \gamma) + \delta^2$	$(\delta - \gamma) + \delta^2$
G_4	$2(\delta - \gamma)$	$2(\delta - \gamma)$	$2(\delta - \gamma)$

Table 2
Total utilities for graphs with three vertices

Graph	Total utility
G_1	0
G_2	$2(\delta - \gamma)$
G_3	$4(\delta - \gamma) + 2\delta^2$
G_4	$6(\delta - \gamma)$

to both a and b in G_1 is simply the difference $(\delta - \gamma)$, between the benefit and the cost associated with the direct tie. Stating the utilities to the actors in G_3 is only slightly more complicated. For a the utility is $2(\delta - \gamma)$ as a bears costs for both ties. The utility for both b and c is $(\delta - \gamma) + \delta^2$ as they derive benefit (δ^2) from the indirect path of length two between them while bearing the cost of only a single tie. For G_4, all actors have a utility of $2(\delta - \gamma)$ as there are no relevant geodesics of length greater than 1 in this graph with three vertices. All the actor utilities are given in Table 1. The total utility for each of the possible graphs is given in Table 2.

Consider the transformation of G_1 to G_2 where a tie can form between any pair of actors. Consider the formation of the tie between a and b. The change in utility for both a and b is $(\delta - \gamma)$. If $\gamma < \delta$, the utility for both actors would be increased by the formation of the tie and the tie would form. And if $\gamma > \delta$, the tie would not form because each actor's utility would decline. Also, in the graph G_2, if either a or b were considering the deletion of the tie and $\gamma < \delta$, both would have their utilities decreased if the tie was removed. It follows that the tie between a and b would not be removed by either actor. But if the tie did exist and $\gamma > \delta$, this tie would be removed and the null graph would result.

Now consider the transition from G_2 to G_3 with the addition of another tie. This could be between a and c or between b and c. Suppose the tie between a and c is considered by either of these actors. The change in utility for a is $(\delta - \gamma)$ and the change in utility for c is $(\delta - \gamma) + \delta^2$. (The utility for b changes by δ^2 but this actor is not involved in the decision to form a tie[4] between a or c. The gain in utility comes from the indirect path of length 2 between b and c.) If $\gamma < \delta$, then both b and c have increased utility and the tie would be

[4] That other actors benefit or incur losses through ties created elsewhere in the network merits attention. It is quite possible that 'third party' actors can invest resources in relations between other pairs of actors either for the formation or deletion of those ties.

Table 3
Conditions for transitions between graphs, $n = 3$

From	To	Tie	Condition for change of tie	
			First actor	Second actor
G_1	G_2	ab	$\boldsymbol{\gamma < \delta}$	$\boldsymbol{\gamma < \delta}$
G_2	G_3	ac	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2$
G_3	G_4	bc	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^2$

formed. Conversely, if we consider G_3 and a potential decision by either b or c to remove the tie between them, and $\gamma < \delta$, then both actors would suffer a decline in utility. The tie would not be removed.

Finally consider the transition from G_3 to G_4 through the addition of a tie between c and b. The change in utility for both b and c is $(\delta - \gamma) - \delta^2$. Each actor benefits if this change is positive, i.e. if $\gamma < (\delta - \delta^2)$. Under this condition, the structure will evolve from G_3 to G_4. There is no change in the utility for a (who is not involved in the decision). If we consider G_4 and a potential decision to remove a tie, for example, between b and c, and $\gamma < (\delta - \delta^2)$, the tie will not be deleted because the utilities for both actors would decline.

Table 3 shows the conditions for the possible transitions through the addition of a single tie where each row of the table corresponds to a transition between graphs. The third column of the table shows the tie added for the transition[5] and the last two columns give the conditions for the transition to occur from the vantage point of each of the actors. The condition that has to hold for the formation of the tie, through the addition of a line, to occur is bolded. Thus, for the first transition from G_1 to G_2, the added tie is between a and b and for each actor the condition for the transition (net benefit is positive) for both actors is $\gamma < \delta$. For the transition between G_2 and G_3, the added tie is between a and c. From the vantage point of a, there is a gain if $\gamma < \delta$. In contrast, from the vantage point of c, there is a gain[6] if $\gamma < \delta + \delta^2$. If the formation of tie requires a mutual decision where neither actor is worse off, the condition for the transition from G_2 to G_3 is $\gamma < \delta$. For the transition from G_3 to G_4, the condition for both actors is $\gamma < \delta - \delta^2$.

For undirected graphs on three vertices, based on an exhaustive search,we have:

Theorem 2. *For graphs with $n = 3$ vertices the equilibrium structures are*:

1. *For $\gamma > \delta$, the null graph (G_1) results*;
2. *For $(\delta - \delta^2) < \gamma < \delta$, the star graph ($G_3$) results*; *and*

[5] Of course, many different labeled sequences are possible, in general. The first tie formed could be (bc), followed by (ac), followed by (ab). In this paper, attention is confined to transitions between classes of isomorphic graphs. Put differently, we focus on structural types of unlabelled graphs and vertex labels are used for descriptive purposes.

[6] Note that if c can impose the tie unilaterally on a, the condition for the transition from its vantage point is $\gamma < \delta + \delta^2$.

3. *If* $\gamma < (\delta - \delta^2)$, *the complete graph* (G_4) *results*.

The second two statements contradict the corresponding claims of JW (1996: 49) that (with $n = 3$) for $\delta - \delta^2 < \gamma < \delta + \delta^2/2$, the star network is strongly efficient and will therefore form. If $\delta < \gamma < (\delta + \delta^2/2)$ as stated in Theorem 1, for $n = 3$, the addition of the tie between a and c to G_2 would have the impact of increasing the utility of c while lowering the utility[7] of a. For γ in this range, the impact on the utility of a is enough to *prevent* the formation of the tie. (When $(\delta <)\gamma < \delta + \delta^2/2$, the change in $U_a > -\delta^2/2$.) This point appears to have been missed by JW. However, if c could impose the tie, then c benefits from the formation of that tie. For situations like this, the variation in the simulations of Hummon (2000) where actors can impose or remove ties unilaterally has considerable interest. This suggests, for this network and parameters in this range, there will be a dynamic equilibrium of an oscillation between G_2 and G_3 where c attempts to impose the tie and a resists and tries to remove it if $\delta < \gamma < (\delta + \delta^2/2)$. Note that if attention is confined to G_3 and we ask under what conditions $U_T(G_3)$ is positive, we get $\gamma < (\delta + \delta^2/2)$, the JW result.

3. Graphs with four vertices

The set of eleven edge graphs with four vertices are shown in Fig. 3. The inclusion lattice for these graphs is shown in Fig. 4 where the numbers on the left are the counts of edges in the graph. If attention is confined to changes in the graphs that occur through the addition[8] of a single tie, the links between the graphs in Fig. 4 are the only possible changes. The utilities for the four actors in these 11 graphs are shown in Table 4. In this table, each panel contains graphs with the same number of edges. The total utility for each of the possible four-vertex graphs is obtained by summing[9] the actor utilities in the rows of Table 4. These are shown in Table 5 where graphs with the same number of lines are in a common panel.

Consider Fig. 4 and Table 4. The transitions from G_1 to G_2 and G_2 to G_3 are the same, essentially, as the first two transitions as for $n = 3$ and the same conclusions hold. For the transition from G_2 to G_4 with the addition of (cd) in Fig. 4, there is no change in the utilities from a and b. The change in utilities for c and d is $\delta - \gamma$ and for $\gamma < \delta$, both c and d get increased utilities and the tie will be form. If $\gamma > \delta$, the tie will not be formed. The arguments hold in reverse for the possible transition from G_4 to G_2 by deleting a tie. That transition will not occur if $\gamma < \delta$. Table 6 summarizes the conditions for transitions to occur where the determining condition is bolded.

Before proceeding through the lattice in this fashion, we take a slight digression. At face value, it is possible to examine the two transitions out of G_2 to see which is the more likely

[7] Suppose that $\gamma = 0.5$ and $\delta = 0.4$. Then adding the tie (ac) to G_2 would increase U_c by 0.06 while U_a is *decreased* by 0.1.

[8] Deletion of ties would move in the opposite direction in the lattice depicted in Fig. 4.

[9] This puts to one side the problem of interpersonal comparisons of utilities, in an attempt to follow closely the analyses of JW and Hummon.

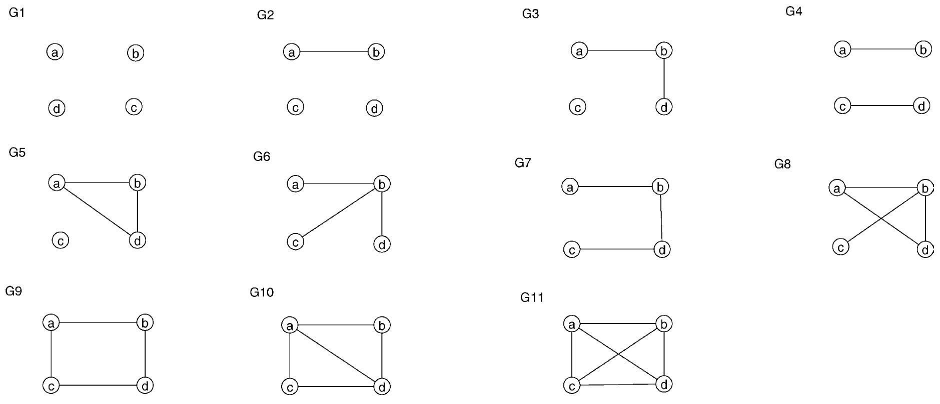

Fig. 3. All edge graphs with four vertices.

Table 4
Actor utilities for all graphs with four vertices

Graph	Actors			
	a	*b*	*c*	*d*
G_1	0	0	0	0
G_2	$(\delta-\gamma)$	$(\delta-\gamma)$	0	0
G_3	$2(\delta-\gamma)$	$(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+\delta^2$	0
G_4	$(\delta-\gamma)$	$(\delta-\gamma)$	$(\delta-\gamma)$	$(\delta-\gamma)$
G_5	$2(\delta-\gamma)$	$2(\delta-\gamma)$	$2(\delta-\gamma)$	0
G_6	$3(\delta-\gamma)$	$(\delta-\gamma)+2\delta^2$	$(\delta-\gamma)+2\delta^2$	$(\delta-\gamma)+2\delta^2$
G_7	$2(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+\delta^2+\delta^3$	$2(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+\delta^2+\delta^3$
G_8	$3(\delta-\gamma)$	$2(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+2\delta^2$
G_9	$2(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2$
G_{10}	$2(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)$	$3(\delta-\gamma)$	$2(\delta-\gamma)+\delta^2$
G_{11}	$3(\delta-\gamma)$	$3(\delta-\gamma)$	$3(\delta-\gamma)$	$3(\delta-\gamma)$

Table 5
Total utilities for graphs with four vertices

Graph	Total utility
G_1	0
G_2	$2(\delta-\gamma)$
G_3	$4(\delta-\gamma)+2\delta^2$
G_4	$4(\delta-\gamma)$
G_5	$6(\delta-\gamma)$
G_6	$6(\delta-\gamma)+6\delta^2$
G_7	$6(\delta-\gamma)+4\delta^2+2\delta^3$
G_8	$8(\delta-\gamma)+4\delta^2$
G_9	$8(\delta-\gamma)+4\delta^2$
G_{10}	$10(\delta-\gamma)+2\delta^2$
G_{11}	$12(\delta-\gamma)$

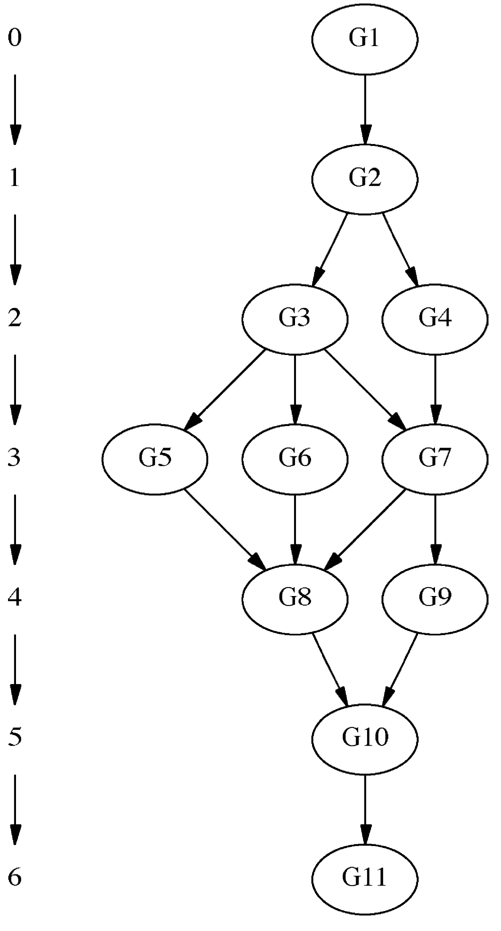

Fig. 4. Lattice of edge graphs with four vertices.

to occur. The basic choice for a is whether or not to form a tie[10] with c and so move the graph to G_3. The increase in a's utility is $\delta - \gamma$ compared with no increase if the choice is made to not form the tie. In G_2, b is structurally equivalent[11] to a and would make the same choice to form a tie with either c or d. For actor c, the choice is to form a tie with a (or b) – and move the graph to G_3 – or form a tie with d and move the structure to G_4. From Table 4, the increase in c's utility when G_3 is formed from G_2 through the formation of a tie between c and a is $(\delta - \gamma) + \delta^2$. Similarly, the utility for c increases by $(\delta - \gamma)$ when a tie is

[10] Structurally, this is exactly the same as considering the formation of a tie with d – as is the case with b considering the formation of a tie with c or with d. Hence the need to consider isomorphism classes of graphs constituting the lattice.

[11] Two actors are structurally equivalent if they are connected to exactly the same other actors in the network. See Lorrain and White (1971) for the foundational statement on structural equivalence and Wasserman and Faust (1994) for a general summary.

Table 6
Conditions for transitions between graphs, $n = 4$

From	To	Tie	Conditions for change of tie	
			Actor 1	Actor 2
G_1	G_2	*ab*	$\boldsymbol{\gamma < \delta}$	$\boldsymbol{\gamma < \delta}$
G_2	G_3	*ac*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2$
G_2	G_4	*cd*	$\boldsymbol{\gamma < \delta}$	$\boldsymbol{\gamma < \delta}$
G_3	G_5	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_3	G_6	*ad*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 2\delta^2$
G_3	G_7	*cd*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2 + \delta^3$
G_4	G_7	*ac*	$\boldsymbol{\gamma < \delta + \delta^2}$	$\boldsymbol{\gamma < \delta + \delta^2}$
G_5	G_8	*ad*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 2\delta^2$
G_6	G_8	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_7	G_8	*ad*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^3$
G_7	G_9	*bd*	$\boldsymbol{\gamma < \delta - \delta^3}$	$\boldsymbol{\gamma < \delta - \delta^3}$
G_8	G_{10}	*cd*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_9	G_{10}	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{10}	G_{11}	*ad*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$

formed with d to give G_4 (if $\gamma < \delta$). The largest utility increase for c is when the transition is made to G_3 from G_2. Exactly the same choice would be made by d. If d forms a tie with a it creates a graph that is isomorphic to G_3. The corresponding row (with G^* denoting a graph isomorphic with G) in Table 4 would be (with the same ordering of vertices):

G_3^*	$2(\delta - \gamma)$	$(\delta - \gamma) + \delta^2$	0	$(\delta - \gamma) + \delta^2$

Given a choice between making a tie with c or making a tie with either a or b, d would have a greater utility increase from choosing an actor other than c. If actors were guided solely by their utilities with full information, it seems the only transition from G_2 is to G_3.

This can be considered in a slightly different way. The above narrative is one that compared the benefits to a between not forming the tie (ac) and forming it. The network (path) created when the tie forms is (ba)(ac). There is another option for a: wait for b and c to establish a tie. When this tie is formed, a's utility moves from $\delta - \gamma$ to $\delta - \gamma + \delta^2$ and a obtains an increase in utility if $\gamma < \delta - \delta^2$. The network (path) created is (ab)(bc). Either way, the star for $n = 3$ is formed. However, the *sequencing* of decisions is important (Hummon, 2000) and if c and d reach an agreement to form a tie first then G_4 is formed.

Here, we return to the lattice in Fig. 4. There are three possible transitions from G_3 in Fig. 4: G_5 is reached from G_3 through the addition of a tie between b and c; G_6 is reached from G_3 through the addition of a tie between a and c; and G_7 is reached from G_3 through the addition of a tie between c and d. The nature of the transition from G_3 to G_5 is essentially the same as the G_3 to G_4 transition for graphs on three vertices. The change in utility for b and c is $\delta - \delta^2 - \gamma$ and their utilities both increase if $\gamma < \delta - \delta^2$. There is no change in the utilities to a and d.

The transition from G_3 to G_6 is the transition to the star for $n = 4$ with the addition of a tie between a and d. The change in utility for a is $(\delta - \gamma)$. Both b and c have an increase in utility of δ^2 through the creation of an additional geodesic of length 2 without the increased cost of creating a tie. Finally, the change in utility for d is $(\delta - \gamma) + \delta^2$. From a's perspective, if $\gamma < \delta$, its utility increases and the tie between a and d will be formed. The transition from G_6 (back) to G_3 will not occur for $\gamma < \delta$. From d's perspective, if $\gamma < \delta + \delta^2$, there is a gain in utility. However, for $\delta < \gamma < \delta + \delta^2$, the formation of the tie benefits d but imposes a cost on a. If ties are formed by mutual agreement, then for γ in this range, the transition will not be made because of the disadvantage imposed on a. If $\gamma > \delta$, the tie between a and d will not be formed and if the tie exists in G_6 it will be eliminated as both a and d would gain from its deletion.

G_3 is transformed to G_7 through the addition of a tie between c and d. With this change, the change in utility for c is $\delta - \gamma$ and the change in d's utility is $(\delta - \gamma) + \delta^2 + \delta^3$. For d, there is an increase in utility if $\gamma < (\delta + \delta^2 + \delta^3)$ and for c, there is a gain in utility if $\gamma < \delta$ implying that the transition to G_7 will occur if $\gamma < \delta$. If, instead, $\delta < \gamma < (\delta + \delta^2 + \delta^3)$, the creation of the tie between c and d benefits d but lowers the utility[12] of c. With mutual agreement required for the formation of ties, this tie would not be created because of the loss of utility to c.[13] The change in utility for a is δ^2 (from the geodesic between a and d that is created). The change in b's utility is δ^3 (from the geodesic of length 3 between b and d).

Structurally, the creation of a tie between b and d creates a graph that is isomorphic the G_7. The utilities in this graph would be:

G_7^*	$2(\delta - \gamma) + \delta^2$	$2(\delta - \gamma) + \delta^2$	$(\delta - \gamma) + \delta^2 + \delta^3$	$(\delta - \gamma) + \delta^2 + \delta^3$

but the same conclusion would be reached regarding the conditions for a transition to G_7.

Given that G_3 can become three different graphs through the addition of a single tie, it is reasonable to digress again to ask about the relative likelihood for these transitions. We need to consider this both from the vantage points of the actors or from the group as a whole. For the latter, from Table 5, the largest total utility comes from G_6 if $\gamma < \delta$. Clearly, $6(\delta - \gamma) + 6\delta^2 > 6(\delta - \gamma)$ as δ^2 is positive. Also, $6(\delta - \gamma) + 6\delta^2 > 6(\delta - \gamma) + 4\delta^2 + 2\delta^3$ because $\delta^2 > \delta^3$ for $0 < \delta < 1$. Thus, the total utility is higher in G_6 than in G_7. The best outcome structure for the group as a whole is G_6. If $\gamma > \delta$ all total utilities would be negative and no ties would form.

With each transition from G_3 there are changes in actor utilities. For a the only decision concerns the transition from G_3 to G_6 with the formation of the tie between a and d (or not). The change in a's utility is $u_a(G_6) - u_a(G_3) = (\delta - \gamma)$. For $\gamma < \delta$, a would choose to form the tie with d. Obviously, if $\gamma < \delta - \delta^2$, the gain in utility to a is greater in G_6 than in G_7. Actor d is faced with a choice between forming a tie with a (leading to G_6) or a tie with c (leading to G_7).[14] For this actor, $u_d(G_6) - u_d(G_3) = (\delta - \gamma) + 2\delta^2$ and $u_d(G_7) - u_d(G_3) = (\delta - \gamma) + \delta^2 + \delta^3$. If it is not the case that $\gamma > \delta + \delta^2 + \delta^3$ (which

[12] For γ in this range, the change in c's utility will satisfy $(\delta - \gamma) > \delta - (\delta + \delta^2 + \delta^3)$ or the change in utility is $> -(\delta^2 + \delta^3)$ and is negative.

[13] If ties can be imposed or removed unilaterally, then d could impose the tie on c. But if ties can also be removed unilaterally, then c would remove the tie at a later point in time. JW allow the unilateral removal of a tie, presumably in cases such as this.

[14] It can also form a tie with b (leading to a graph that is isomorphic to G_7), a change that is considered below.

would make all total utilities negative), $u_d(G_6) - u_d(G_3) > u_d(G_7) - u_d(G_3)$ because[15] $\delta^2 > \delta^3$ with $0 < \delta < 1$. The best choice for d is forming a tie with a to move the structure to G_6. In G_3, the choice facing b is whether to form the tie with c (or not) and move the graph to G_5 or to form a tie with d and create a graph that is isomorphic to G_7. For the former, the change in utility for b, $u_b(G_5) - u_b(G_3) = \delta - \gamma - \delta^2$. For $\gamma < \delta$, this difference is negative and the tie would not be formed but for $\gamma < \delta - \delta^2$ the tie is formed. Another option for b is to wait for the formation of (ad) and have a utility gain of δ^2 (in G_6). But if $\gamma < \delta - \delta^2$, the gain is less than when a tie with d is formed. As b and c are structurally identical, the choices for c are the same as for b and G_7 the preferred structure for c also. If a and d get to choose first in G_3, the transition to G_6 will occur and if c and b get to choose first, the transition to G_7 will be selected. It would seem that the transition to G_5 would not occur if actors choose ties to increase their utilities. Clearly, the sequence of decision making becomes critical in empirical situations for the determining the nature of a transition to another graph.

There is an additional complication for considering moves from G_3 to graphs isomorphic with G_7. Consider the graph with the symmetric ties $\{(ab), (ac), (bd)\}$, that comes from the formation of (bd), and G_7 with {(ab), (ac), (cd)}. The utility to c in this alternative graph is $u_c(G_7^{\text{alt}}) = (\delta - \gamma) + \delta^2 + \delta^3$ compared to the utility $u_c(G_7) = 2(\delta - \gamma) + \delta^2$ in G_7. The difference in these utilities is $u_c(G_7^{\text{alt}}) - u_c(G_7) = \delta - \gamma + \delta^3$. For $\gamma < \delta + \delta^3$, this difference is positive. An interpretation of this difference is that c has greater utility when the tie (bd) is added to G_3 than when (cd) is added to G_3. Its utility is increased by not having to invest in the (cd) tie while benefiting from the path to d through a and b which requires investments from a and b. Similarly, b would prefer having c invest in the (cd) tie instead of b investing in the (bd) tie if $\gamma < \delta + \delta^3$.

With the transition from G_4 to G_7, the change in the utility of a and c is $(\delta - \gamma) + \delta^2$ which is an increase if $\gamma < \delta + \delta^2$. The change in utility for b and d is $\delta^2 + \delta^3$ which is positive. Thus, the transition will be made[16] if $\gamma < \delta + \delta^2$. The transition from G_5 to G_8 involves the addition of a tie between d and a[17] with a change in the utility to a of $\delta - \gamma$. The condition, $\gamma < \delta$ means this change in utility for a is positive. Both b and c have an increase in utility of δ^2 which is necessarily positive. For d, the change in utility is $(\delta - \gamma) + 2\delta^2$. So, for d, this change is positive if $\gamma < \delta + 2\delta^2$. However, with this condition, the change in utility for a satisfies $\delta - \gamma > -2\delta^2$. For the condition $\delta < \gamma < \delta + 2\delta^2$, a suffers a decline in utility and the tie between a and d would not be formed. Under the constraint that the formation of ties benefits both actors, this tie would not be formed.

When a tie forms between b and c in G_6 we get G_8. The change in utility for b and c is $(\delta - \gamma) - \delta^2$. This change benefits them when $\gamma < (\delta - \delta^2)$ and the tie between b and c

[15] More directly, $u_d(G_6) > u_d(G_7)$.

[16] Note that if $\gamma < \delta + \delta^2 + \delta^3$, the transition leads to both an increase in the utility for both b and d and a reduction of a's utility because $(\delta - \gamma) + \delta^2 > -\delta^3$ and therefore can be negative. However, γ in this range prevents other transitions and this would not occur. Even so, the idea of changes in one part of a network that are advantageous for two actors but affect other actors adversely is worth examining – as noted by both JW and Hummon.

[17] Either the addition of a tie between d and b or the addition of a tie between d and c creates a graph that is isomorphic with G_8. It follows that working out the details for either of these transitions will lead to the same conclusions. Only the labeling of the vertices would change. There is only one structure that can be generated from G_5.

will form.[18] Neither a nor d is involved in this decision and as the change in utility for them is zero, they are not adversely affected by the change.

As drawn in Fig. 3, there is no change from G_7 leading to G_8 where the labeling is preserved. However, if a tie between a and d is added (or between b and c) a graph results that is isomorphic with G_8. Suppose it is the tie between a and d. The row of utilities to the actors is (for a, b, c and d, respectively):

G_8^*	$3(\delta-\gamma)$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+\delta^2$

The only difference between this and the row labeled G_8 in Table 4 is the interchange of b and d. The difference in utility for a is $(\delta-\gamma)-\delta^2$ and a's utility is increased if $\gamma<\delta-\delta^2$. The difference in utility for b is $\delta^2-\delta^3$ which is necessarily positive. The difference in utility for c is 0 so this actor is not adversely affected by the change. Finally, the change in utility for d is $(\delta-\gamma)-\delta^3$ so d's utility is increased if $\gamma<\delta-\delta^3$. If $\delta-\delta^2<\gamma<\delta-\delta^3$, the change in a's utility is $(\delta-\gamma)-\delta^2<\delta^3-\delta^2$ which is negative given the range of δ. With a's utility adversely affected with γ in this range, and mutuality is needed for the forging of ties, the transition from G_7 to G_8 occurs only if $\gamma<\delta-\delta^2$.

The transition between G_7 and G_9 involves the addition of a tie between b and d. The change in the utility for both b and d is $2(\delta-\gamma)+\delta^2-((\delta-\gamma)+\delta^2+\delta^3)$. This is $(\delta-\gamma)-\delta^3$ and the change is positive when $\gamma<\delta-\delta^3$. Both a and c are unaffected by this transition as the difference in utility for them is 0.

A graph, G_{10}^*, isomorphic with G_{10} can be reached from both G_8 and G_9. The transition from G_8 to G_{10}^* involves adding a tie between c and d to create a graph that is isomorphic with G_{10}. The comparison for change in utilities comes from:

G_{10}^*	$3(\delta-\gamma)$	$2(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)$	$2(\delta-\gamma)+\delta^2$

The change in utility for c and d is $(\delta-\gamma)-\delta^2$ and this transition will occur if $\gamma<\delta-\delta^2$. The utilities to a and d are not affected by the change.

The transition from G_9 to G_{10} is through adding a tie between b and c. For both b and c the change in utility is $3(\delta-\gamma)-(2(\delta-\gamma)+\delta^2)$ which reduces to $(\delta-\gamma)-\delta^2$. The change will occur if $\gamma<\delta-\delta^2$. As the change in utility for both a and d is 0, they are unaffected by the change.

The last transition shown in Fig. 4 is from G_{10} to G_{11} with the addition of a tie between b and d. As the change in utility for both b and d is $(\delta-\gamma)-\delta^2$, the transition occurs for $\gamma<\delta-\delta^2$. The utilities for a and d are unaffected by this change. Note that for the denser networks (after G_6 has been reached), there is a single condition[19] for tie changes, $\gamma<\delta-\delta^2$.

Table 6 shows the above conditions for transitions to occur between the graphs in the lattice of graphs shown in Fig. 4. If $\gamma<\delta-\delta^2$, every transition in the table can occur and

[18] Note that adding a tie between b and d creates a graph that is isomorphic with G_8 as does the addition of a tie between c and d. The same arguments hold for each graph as the only difference is found in their labeling.

[19] An exception is for the transition between G_7 and G_9 where the condition is $\gamma<\delta-\delta^3$. Of course, if $\gamma<\delta-\delta^2$ this transition would occur.

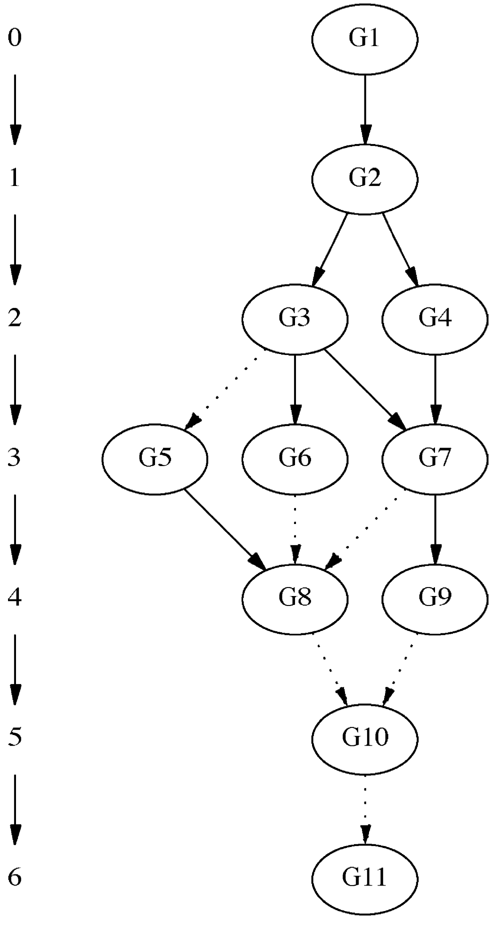

Fig. 5. Transitions for $\delta - \delta^2 < \gamma < \delta - \delta^3$ in four-vertex edge graphs.

the complete graph is the stable structure that is reached. The next range of values for γ that we consider is $\delta - \delta^2 < \gamma < \delta - \delta^3$. Fig. 5 shows, in the lattice of Fig. 4, which transitions that are possible (with a solid line) and the transitions which are not possible (with a dotted line). For example, the transitions to G_3 and G_4 can occur but G_5 could not be reached from G_3 (because this transition requires $\gamma < \delta - \delta^2$). It follows that the transition between G_5 and G_8 could not occur (despite the solid line linking these graphs in Fig. 5). Both G_6 (the star for $n = 3$) and G_7 could be reached from G_3 (and G_7 can be reached from G_4). However, the transition out from G_6 to G_8 cannot occur with $\delta - \delta^2 < \gamma$. From G_7 only the transition to G_9 (the cycle for $n = 4$) from G_7 can occur. The two stable structures that can be reached with the range for γ are the star and the cycle. The cycle is one of the structures that could be a stable equilibrium that were identified by Hummon (2000). Now consider $\delta - \delta^3 < \gamma < \delta$. The transitions to G_3 and G_4 can both occur. From G_3 both G_6 and G_7 can be reached. (G_7 can also be reached from G_4.) No transitions are possible from G_6 and

G_7 for this parameter range for γ. Thus, the star (G_6) is stable as well as G_7 (which can be described as a 'near cycle' in the language of Hummon (2000).) Finally, if $\gamma > \delta$ there are no ties and the null graph results.

From Table 6 for undirected graphs on four vertices, we have:

Theorem 3. *For graphs with* $n = 4$ *vertices, the equilibrium structures are*:

1. *For* $\gamma > \delta$, *the null graph* (G_1) *results*;
2. *For* $\delta > \gamma > (\delta - \delta^3)$, *the star* ($G_6$) *and the near cycle* ($G_7$) *result*;
3. *For* $(\delta - \delta^3) > \gamma > (\delta - \delta^2)$, *the star* ($G_6$) *and cycle* ($G_9$) *result*; *and*
4. *If* $\gamma < (\delta - \delta^2)$, *the complete graph results*.

The second and third parts of this description provide an alternative to Theorem 1 by breaking the range $\delta > \gamma > \delta - \delta^2$ into two part and shows that, in addition to the star graph, both the cycle and the near cycle can be stable for particular ranges of γ, consistent with the results of Hummon (2000).

These transitions to stable structures can be viewed also in terms of the total utility $u_T(G_i)$. For $\delta > \gamma > \delta - \delta^3$, there are two stable structures. One is the star (G_6) and the near-cycle (G_7). For $\gamma = \delta$, we have $u_T(G_6) = 6\delta^2$ and $u_T(G_7) = 4\delta^2 + 2\delta^3$. Clearly, $u_T(G_6) > u_T(G_7)$ because $\delta^2 > \delta^3$ for $0 < \delta < 1$. At the upper bound, the total utility of the star is higher than is the utility for the near-cycle. For $\gamma = \delta - \delta^3$, $u_T(G_6) = 6\delta^2 + 6\delta^3$ and $u_T(G_7) = 4\delta^2 + 8\delta^3$ and, again, $u_T(G_6) > u_T(G_7)$ at the lower bound for this range of γ. Throughout the range of $\delta > \gamma > \delta - \delta^3$, the star has higher total utility. Recall that if the graph is at G_3, if *a* or *d* choose to form the tie (*ad*), the graph moves to G_6 and if *b* or *c* forms (*bd*) or (*cd*), the graph[20] moves to G_7. This is a fateful choice for the actors, because when G_7 is reached, moving back to G_3 is no longer possible. The near-cycle becomes a sub-optimal but stable network structure when $\delta > \gamma > \delta - \delta^3$. One key difference between Theorem 1 and the results from the approach taken here is that the former is based on graph utilities while the latter is based on a detailed examination of possible transitions. The two approaches need not generate the same results. Note that $U_T(G_6) > 0$, if $\gamma < \delta + \delta^2$, the JW result yielding a star for $n = 4$ (although in this range for γ some transitions needed to reach G_6 cannot occur).

For $(\delta - \delta^3) > \gamma > \delta - \delta^2$, there are also two stable structures – the star (G_6) and the cycle (G_9). For the star when $\gamma = \delta - \delta^3$, as detailed above, $u_T(G_6) = 6\delta^2 + 6\delta^3$. The total utility for the cycle is $u_T(G_9) = 4\delta^2 + 8\delta^3$. For this value of γ, $u_T(G_6) - u_T(G_9) = 2\delta^2 - 2\delta^3 > 0$ which makes the cycle sub-optimal (but stable). However, at the lower point of this range for γ, i.e. when $\gamma = \delta - \delta^2$, $u_T(G_6) = u_T(G_9) = 12\delta^2$. For this value of γ, both the star and the cycle have the same total utility. Again, the message over most of this range is the cycle can be stable even though it is sub-optimal. We note that, in a sense, the definition of equilibrium has been generalized in this discussion. For JW the strongly efficient edge graphs are defined as equilibria. In this discussion, as well as in Hummon (2000), a graph is an equilibrium structure if it can be reached but no transitions out of it are

[20] The actor *d* gains in utility with this transition and would agree to form one of these two ties.

possible given the regime of parameters for the process. It is stable because no change is possible unless the parameters change even though such equilibria can be – and sometimes are – suboptimal.

4. Graphs with five vertices

There are 34 graphs with five vertices. These are shown in Fig. 6. The lattice of all five-vertex graphs is shown in Fig. 7 where the link between two graphs in the lattice represents the addition of one edge. Again, the number of edges for the graphs in each row are shown on the left of this figure.

In exactly the same fashion for the three- and four-vertex graphs considered above, it is straightforward to determine the utility for each actor in these graphs. These utilities are shown in Table 7 (and the total utilities are given later in Table 10).

Also in parallel with the three- and four-vertex graphs, the conditions for a tie to be added to a graph can be determined. Each transition is added by mutual consent depending on the outcome for each actor in terms of utilities. For each of the transitions in the lattice of Fig. 7, the conditions for the change (under which the actor's utility is increased by the addition of the tie) for each actor are given in Tables 8 and 9. For some of the transitions, the labeling

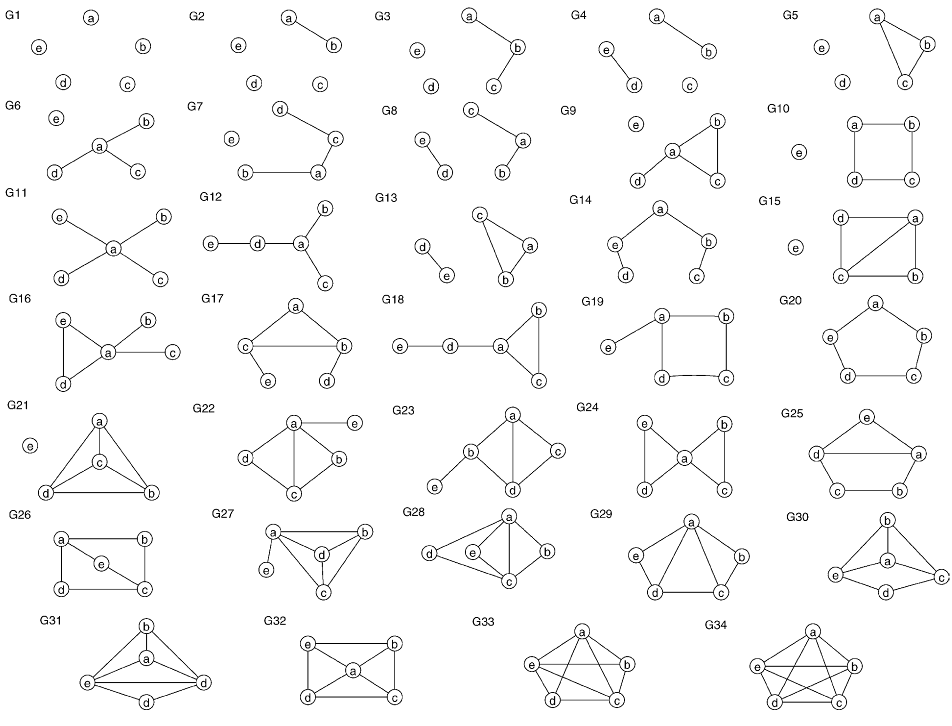

Fig. 6. Edge graphs with five vertices.

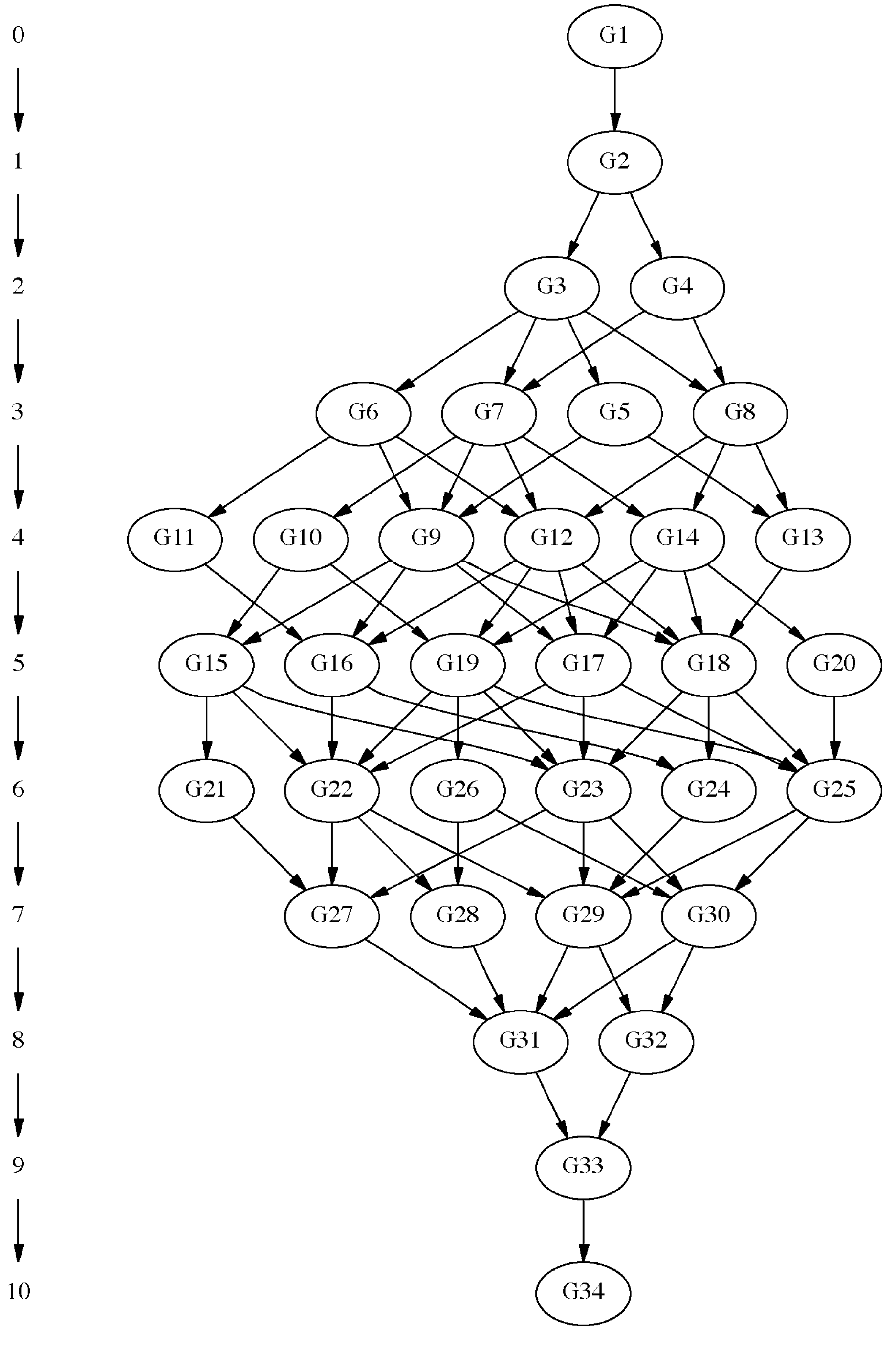

Fig. 7. Lattice of graphs with five vertices.

Table 7
Actor utilities for all graphs with five vertices

Graph	Actors				
	a	b	c	d	e
G_1	0	0	0	0	0
G_2	$(\delta-\gamma)$	$(\delta-\gamma)$	0	0	0
G_3	$2(\delta-\gamma)$	$(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+\delta^2$	0	0
G_4	$(\delta-\gamma)$	$(\delta-\gamma)$	$(\delta-\gamma)$	$(\delta-\gamma)$	0
G_5	$2(\delta-\gamma)$	$2(\delta-\gamma)$	$2(\delta-\gamma)$	0	0
G_6	$3(\delta-\gamma)$	$(\delta-\gamma)+2\delta^2$	$(\delta-\gamma)+2\delta^2$	$(\delta-\gamma)+2\delta^2$	0
G_7	$2(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+\delta^2+\delta^3$	$2(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+\delta^2+\delta^3$	0
G_8	$2(\delta-\gamma)$	$(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+\delta^2$	$(\delta-\gamma)$	$(\delta-\gamma)$
G_9	$3(\delta-\gamma)$	$2(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+2\delta^2$	0
G_{10}	$2(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2$	0
G_{11}	$4(\delta-\gamma)$	$(\delta-\gamma)+3\delta^2$	$(\delta-\gamma)+3\delta^2$	$(\delta-\gamma)+3\delta^2$	$(\delta-\gamma)+3\delta^2$
G_{12}	$3(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+2\delta^2+\delta^3$	$(\delta-\gamma)+2\delta^2+\delta^3$	$2(\delta-\gamma)+2\delta^2$	$(\delta-\gamma)+\delta^2+2\delta^3$
G_{13}	$2(\delta-\gamma)$	$2(\delta-\gamma)$	$2(\delta-\gamma)$	$(\delta-\gamma)$	$(\delta-\gamma)$
G_{14}	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+\delta^2+\delta^3$	$2(\delta-\gamma)+\delta^2+\delta^3$	$(\delta-\gamma)+\delta^2+\delta^3+\delta^4$	$(\delta-\gamma)+\delta^2+\delta^3+\delta^4$
G_{15}	$3(\delta-\gamma)$	$2(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)$	$2(\delta-\gamma)+\delta^2$	0
G_{16}	$4(\delta-\gamma)$	$(\delta-\gamma)+3\delta^2$	$(\delta-\gamma)+3\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$
G_{17}	$2(\delta-\gamma)+2\delta^2$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+2\delta^2+\delta^3$	$(\delta-\gamma)+2\delta^2+\delta^3$
G_{18}	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2+\delta^3$	$2(\delta-\gamma)+\delta^2+\delta^3$	$2(\delta-\gamma)+2\delta^2$	$(\delta-\gamma)+\delta^2+2\delta^3$
G_{19}	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+\delta^2+\delta^3$	$(\delta-\gamma)+2\delta^2+\delta^3$
G_{20}	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$
G_{21}	$3(\delta-\gamma)$	$3(\delta-\gamma)$	$3(\delta-\gamma)$	$3(\delta-\gamma)$	0
G_{22}	$4(\delta-\gamma)$	$2(\delta-\gamma)+2\delta^2$	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+2\delta^2$	$(\delta-\gamma)+3\delta^2$
G_{23}	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+\delta^2+\delta^3$	$(\delta-\gamma)+2\delta^2+\delta^3$
G_{24}	$4(\delta-\gamma)$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$
G_{25}	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$
G_{26}	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+2\delta^2$
G_{27}	$4(\delta-\gamma)$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$(\delta-\gamma)+3\delta^2$
G_{28}	$4(\delta-\gamma)$	$2(\delta-\gamma)+2\delta^2$	$4(\delta-\gamma)$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$
G_{29}	$4(\delta-\gamma)$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+2\delta^2$	$2(\delta-\gamma)+2\delta^2$
G_{30}	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$2(\delta-\gamma)+2\delta^2$
G_{31}	$3(\delta-\gamma)+\delta^2$	$4(\delta-\gamma)$	$3(\delta-\gamma)+\delta^2$	$4(\delta-\gamma)$	$2(\delta-\gamma)+2\delta^2$
G_{32}	$4(\delta-\gamma)$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$	$3(\delta-\gamma)+\delta^2$
G_{33}	$4(\delta-\gamma)$	$4(\delta-\gamma)$	$3(\delta-\gamma)+\delta^2$	$4(\delta-\gamma)$	$3(\delta-\gamma)+\delta^2$
G_{34}	$4(\delta-\gamma)$	$4(\delta-\gamma)$	$4(\delta-\gamma)$	$4(\delta-\gamma)$	$4(\delta-\gamma)$

of the edges in the graphs in Fig. 6 can be used directly. For example, the transition from G_3 to G_5 comes with the addition of the edge (*bc*). For the transition from G_4 to G_7, the labeling in Fig. 6 does not work in this fashion. One way of making it work is to add the link (*be*) to create a graph isomorphic with G_7. Another, is to consider the graph with (*ab*) and (*cd*) – which is isomorphic with G_4 – and add (*ac*). This is shown in row 8 of Table 8. In each such case, the utilities in Table 7 are used (but are permuted) when isomorphic destination graphs are used. These transitions are marked by using square parentheses in the edge addition columns of Tables 8 and 9. From these tables it is straightforward to determine when transitions can occur. The bolded entries mark the conditions under which the tie is created, again assuming that each actor has to benefit from the created tie. Four broad intervals are considered for γ:

1. $\gamma < \delta - \delta^2$;
2. $\delta - \delta^2 < \gamma < \delta - \delta^3$;
3. $\delta - \delta^3 < \gamma < \delta$;
4. $\delta < \gamma$.

For $\gamma < \delta - \delta^2$, all transitions are possible and the complete graph results, consistent with JW. For the range of values for γ given by $\delta - \delta^2 < \gamma < \delta - \delta^3$, some interesting results emerge. Clearly, in this range, the condition of $\gamma < \delta - \delta^2$ is not met and all of the transitions requiring this will not occur. The possible transitions are shown in Fig. 8 by the solid lines. The dotted lines show transitions that cannot occur once $\gamma > \delta - \delta^2$.

There are four graphs that appear as 'terminus' graphs (in the sense of being reached by transitions in the lattice but where no further transitions are possible by adding an edge) when $\delta - \delta^2 < \gamma < \delta - \delta^3$. There are solid lines into them but there are no solid lines leading out of them. These graphs are G_{11}, G_{17}, G_{20}, and G_{26}. (Although there is a solid line into G_{16} it is not a terminal graph because the transition path in the lattice goes through G_5 and this graph cannot be reached from G_3 for the range of γ considered here. The same holds for G_{25} because there are only two solid paths from the unreachable G_5. And the solid path into G_{30} starts at G_{15}, a graph that cannot be reached from G_9 nor from G_{10} for this range of γ.)

The graph G_{11} is the star, one of the structures anticipated by Jackson and Wolinsky (1996). The cycle (G_{20}) is another stable structure and is one of the structures identified by Hummon (2000). G_{17} can be viewed as a near-cycle and was anticipated by Hummon. However, G_{26} seems a new structure that if reached is stable – but it would appear to rare given the Hummon simulations.

We next consider γ in the range $(\delta - \delta^3) < \gamma < \delta$ and the permitted transitions are shown with solid lines in Fig. 9. As the range of values has been shifted upwards for γ, all of the transitions precluded in Fig. 8 are precluded in Fig. 9. In addition, the following transitions[21] cannot occur for $\gamma > \delta - \delta^3$: G_7 to G_{10}; G_{12} to G_{19}; G_{14} to G_{17}; G_{19} to G_{26} and G_{23} to G_{30}. The set of possible terminal graphs is G_{11}, G_{12}, G_{19} and G_{20}.

[21] The transitions G_{18} to G_{25} and G_{23} to G_{30} are eliminated – but for each transition,the first graph cannot be reached for γ in this range.

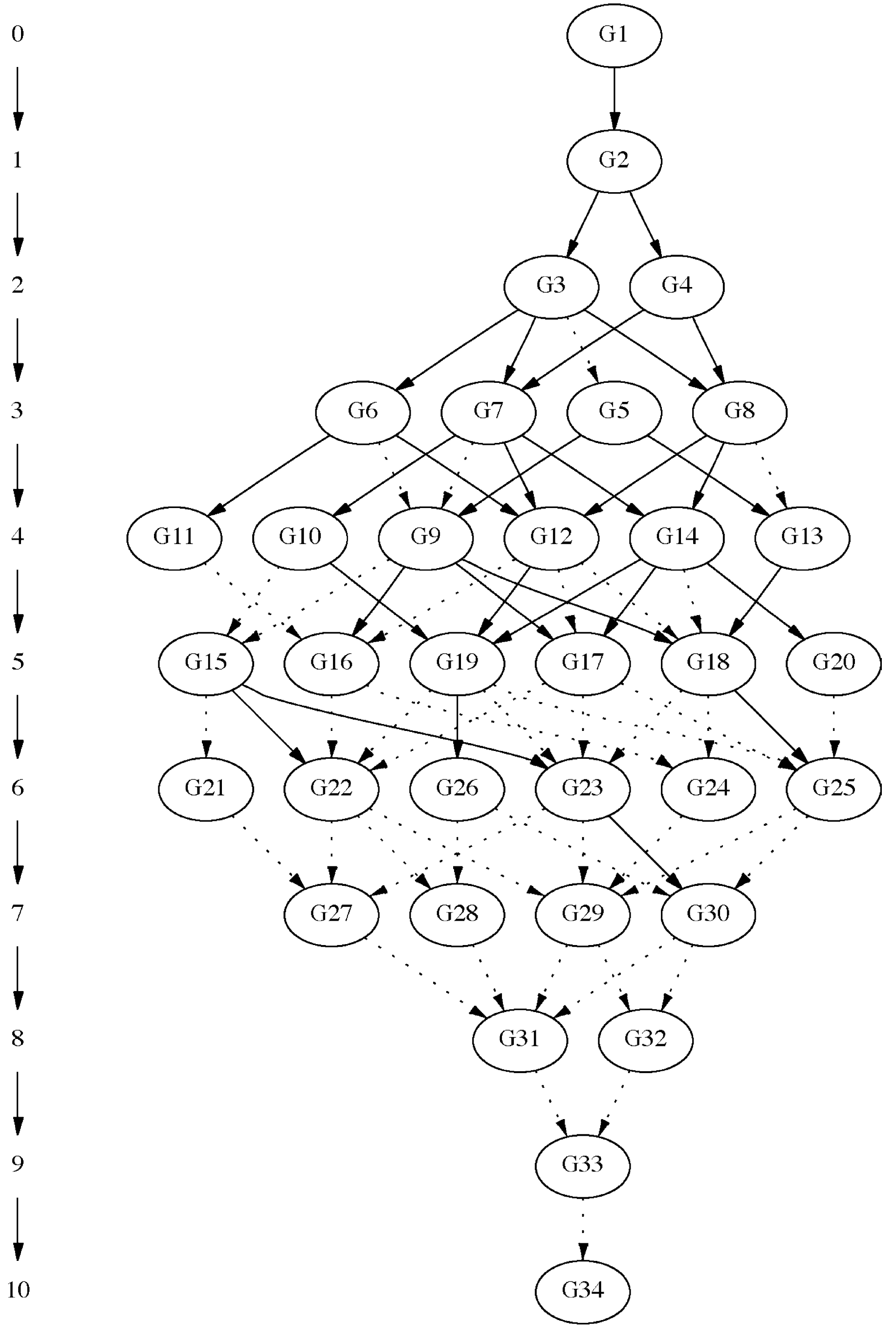

Fig. 8. Transitions between five-vertex graphs for $\delta - \delta^2 < \gamma < \delta - \delta^3$.

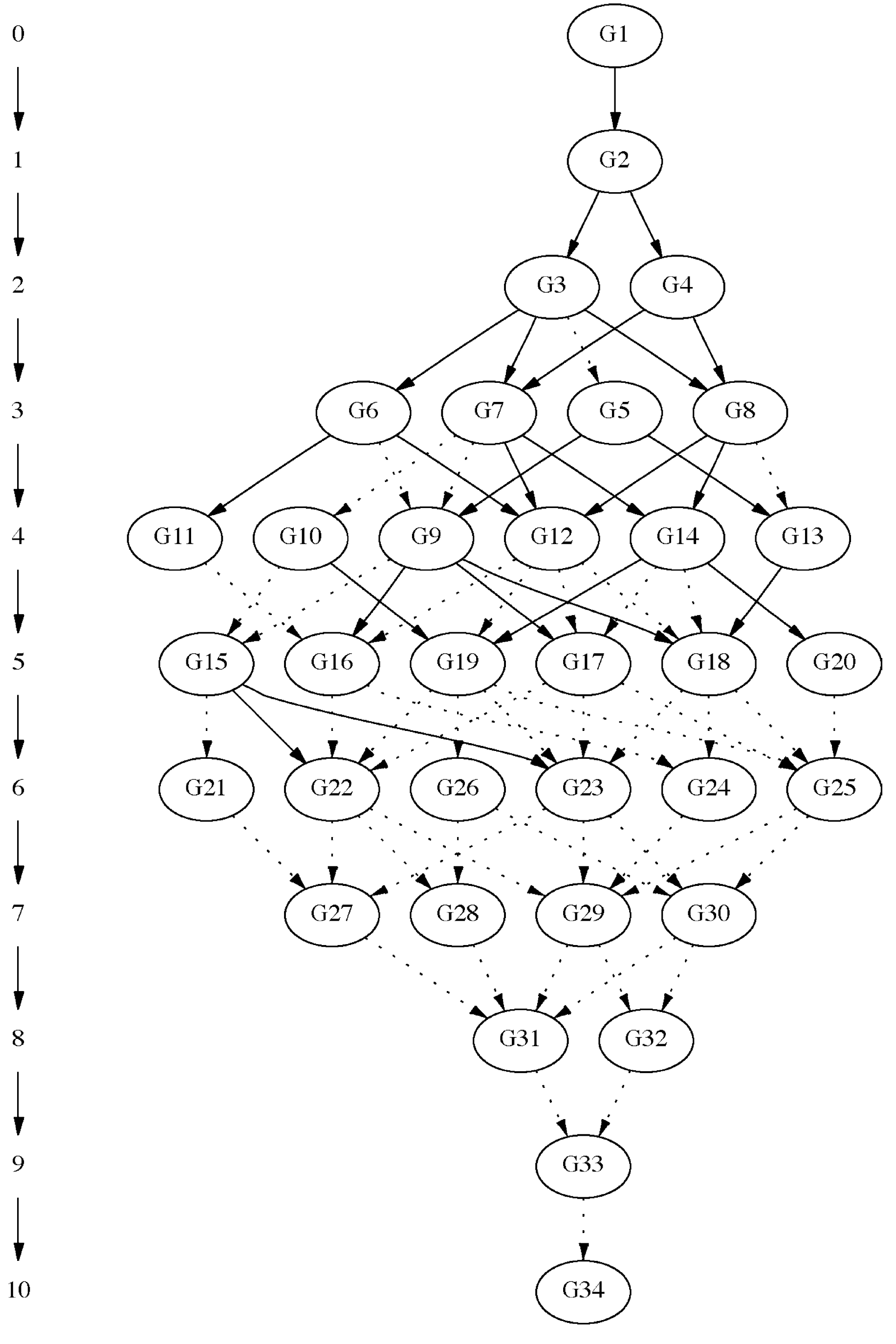

Fig. 9. Transitions between five-vertex graphs for $\delta - \delta^3 < \gamma < \delta$.

Table 8
Conditions for transitions into graphs G_1 through G_{20}, $n = 5$

From	To	Tie	Conditions for tie	
			Actor 1	Actor 2
G_1	G_2	*ab*	$\boldsymbol{\gamma < \delta}$	$\boldsymbol{\gamma < \delta}$
G_2	G_3	*ab*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2$
G_2	G_4	*de*	$\boldsymbol{\gamma < \delta}$	$\boldsymbol{\gamma < \delta}$
G_3	G_5	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_3	G_6	*ad*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 2\delta^2$
G_3	G_7	*cd*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2 + \delta^3$
G_3	G_8	*de*	$\boldsymbol{\gamma < \delta}$	$\boldsymbol{\gamma < \delta}$
G_4	G_7	[*ac*]	$\boldsymbol{\gamma < \delta + \delta^2}$	$\boldsymbol{\gamma < \delta + \delta^2}$
G_4	G_8	[*ac*]	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2$
G_5	G_9	*ad*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 2\delta^2$
G_5	G_{13}	*ed*	$\boldsymbol{\gamma < \delta}$	$\boldsymbol{\gamma < \delta}$
G_6	G_9	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_6	G_{11}	*ea*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 3\delta^2$
G_6	G_{12}	*ed*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2 + 2\delta^3$
G_7	G_9	[*ad*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^3$
G_7	G_{10}	*bd*	$\boldsymbol{\gamma < \delta - \delta^3}$	$\boldsymbol{\gamma < \delta - \delta^3}$
G_7	G_{12}	[*ae*]	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 2\delta^2 + \delta^3$
G_7	G_{14}	*be*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2 + \delta^3 + \delta^4$
G_8	G_{12}	*ad*	$\boldsymbol{\gamma < \delta + \delta^2}$	$\gamma < \delta + 2\delta^2$
G_8	G_{13}	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_8	G_{14}	[*be*]	$\boldsymbol{\gamma < \delta + \delta^2}$	$\gamma < \delta + \delta^2 + \delta^3$
G_9	G_{15}	*cd*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_9	G_{16}	[*ae*]	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 3\delta^2$
G_9	G_{17}	[*be*]	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 2\delta^2 + \delta^3$
G_9	G_{18}	*de*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + \delta^2 + 2\delta^3$
G_{10}	G_{15}	[*ad*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{10}	G_{19}	*ae*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 2\delta^2 + \delta^3$
G_{11}	G_{16}	*de*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{12}	G_{16}	*ae*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta + \delta^2 - 2\delta^3$
G_{12}	G_{17}	[*bd*]	$\gamma < \delta - \delta^3$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{12}	G_{18}	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{12}	G_{19}	[*ce*]	$\boldsymbol{\gamma < \delta - \delta^3}$	$\boldsymbol{\gamma < \delta - \delta^3}$
G_{13}	G_{18}	*ad*	$\boldsymbol{\gamma < \delta + \delta^2}$	$\gamma < \delta + 2\delta^2$
G_{14}	G_{17}	*bc*	$\boldsymbol{\gamma < \delta - \delta^3}$	$\boldsymbol{\gamma < \delta - \delta^3}$
G_{14}	G_{18}	[*ad*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^4$
G_{14}	G_{19}	[*bd*]	$\boldsymbol{\gamma < \delta - \delta^3}$	$\gamma < \delta + \delta^2 - \delta^3 - \delta^4$
G_{14}	G_{20}	*ed*	$\boldsymbol{\gamma < \delta + \delta^2 - \delta^3 - \delta^4}$	$\boldsymbol{\gamma < \delta + \delta^2 - \delta^3 - \delta^4}$

In summary, we have for undirected graphs on five vertices, we have:

Theorem 4. *For graphs with $n = 5$ vertices, the equilibrium structures are*:

1. *For $\gamma > \delta$, the null graph (G_1) results*;
2. *For $\delta > \gamma > \delta - \delta^3$, the star ($G_{11}$), the cycle ($G_{20}$), G_{12} and G_{19} result*;

Table 9
Conditions for transitions into graphs G_{21} through G_{34}, $n = 5$

From	To	Tie	Conditions for change of tie	
			Actor 1	Actor 2
G_{15}	G_{21}	*bd*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{15}	G_{22}	*ae*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 3\delta^2$
G_{15}	G_{23}	*be*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 2\delta^2 + \delta^3$
G_{16}	G_{22}	[*cd*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{16}	G_{24}	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{17}	G_{22}	[*bd*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^3$
G_{17}	G_{23}	[*ad*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{17}	G_{25}	[*ed*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{18}	G_{23}	[*cd*]	$\gamma < \delta - \delta^3$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{18}	G_{24}	*ae*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta + \delta^2 - 2\delta^3$
G_{18}	G_{25}	*bc*	$\boldsymbol{\gamma < \delta - \delta^3}$	$\gamma < \delta + \delta^2 - 2\delta^3$
G_{19}	G_{22}	[*ad*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^3$
G_{19}	G_{23}	[*bc*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{19}	G_{25}	[*ce*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^3$
G_{19}	G_{26}	[*de*]	$\boldsymbol{\gamma < \delta - \delta^3}$	$\boldsymbol{\gamma < \delta - \delta^3}$
G_{20}	G_{25}	*bc*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{21}	G_{27}	*ae*	$\boldsymbol{\gamma < \delta}$	$\gamma < \delta + 3\delta^3$
G_{22}	G_{27}	*bd*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{22}	G_{28}	*ce*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{22}	G_{29}	*be*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{23}	G_{27}	[*bd*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^3$
G_{23}	G_{29}	*ae*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\gamma < \delta - \delta^3$
G_{23}	G_{30}	*de*	$\boldsymbol{\gamma < \delta - \delta^3}$	$\boldsymbol{\gamma < \delta - \delta^3}$
G_{24}	G_{29}	*be*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{25}	G_{29}	[*ae*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{25}	G_{30}	*cd*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{26}	G_{28}	*ad*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{26}	G_{30}	[*bc*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{27}	G_{31}	[*be*]	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{28}	G_{31}	*be*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{29}	G_{31}	*ce*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{29}	G_{32}	*de*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{30}	G_{31}	*bd*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{30}	G_{32}	*ae*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{31}	G_{33}	*ae*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{32}	G_{33}	*bd*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$
G_{33}	G_{34}	*ce*	$\boldsymbol{\gamma < \delta - \delta^2}$	$\boldsymbol{\gamma < \delta - \delta^2}$

3. *For* $(\delta - \delta^3) > \gamma > \delta - \delta^2$, *the star* ($G_{11}$), *the cycle* ($G_{20}$), G_{17}, *and* G_{26} *result*; *and*
4. *If* $\gamma < (\delta - \delta^2)$, *the complete graph results.*

Table 10 has the total utilities for each of the 34 five-edge graphs in Table 7. Among the graphs with four edges, $u_T(G_{11}) = 8(\delta - \gamma) + 12\delta^2$ has the highest utility for $0 < \delta < 1$. Regardless of the values of δ and γ, this total utility is higher than the total utilities for all graphs with fewer edges. Among the graphs with five edges, $u_T(G_{16}) = u_T(G_{20}) =$

Table 10
Total utilities for graphs with five vertices

Graph	Total utility
G_1	0
G_2	$2(\delta - \gamma)$
G_3	$4(\delta - \gamma) + 2\delta^2$
G_4	$4(\delta - \gamma)$
G_5	$6(\delta - \gamma)$
G_6	$6(\delta - \gamma) + 6\delta^2$
G_7	$6(\delta - \gamma) + 4\delta^2 + 2\delta^3$
G_8	$6(\delta - \gamma) + 2\delta^2$
G_9	$8(\delta - \gamma) + 4\delta^2$
G_{10}	$8(\delta - \gamma) + 4\delta^2$
G_{11}	$8(\delta - \gamma) + 12\delta^2$
G_{12}	$8(\delta - \gamma) + 8\delta^2 + 4\delta^3$
G_{13}	$8(\delta - \gamma)$
G_{14}	$8(\delta - \gamma) + 6\delta^2 + 4\delta^3 + 2\delta^4$
G_{15}	$10(\delta - \gamma) + 2\delta^2$
G_{16}	$10(\delta - \gamma) + 10\delta^2$
G_{17}	$10(\delta - \gamma) + 8\delta^2 + 2\delta^3$
G_{18}	$10(\delta - \gamma) + 6\delta^2 + 4\delta^3$
G_{19}	$10(\delta - \gamma) + 8\delta^2 + 2\delta^3$
G_{20}	$10(\delta - \gamma) + 10\delta^2$
G_{21}	$12(\delta - \gamma)$
G_{22}	$12(\delta - \gamma) + 8\delta^2$
G_{23}	$12(\delta - \gamma) + 6\delta^2 + 2\delta^3$
G_{24}	$12(\delta - \gamma) + 8\delta^2$
G_{25}	$12(\delta - \gamma) + 8\delta^2$
G_{26}	$12(\delta - \gamma) + 8\delta^2$
G_{27}	$14(\delta - \gamma) + 6\delta^2$
G_{28}	$14(\delta - \gamma) + 6\delta^2$
G_{29}	$14(\delta - \gamma) + 6\delta^2$
G_{30}	$14(\delta - \gamma) + 6\delta^2$
G_{31}	$16(\delta - \gamma) + 4\delta^2$
G_{32}	$16(\delta - \gamma) + 4\delta^2$
G_{33}	$18(\delta - \gamma) + 2\delta^2$
G_{34}	$20(\delta - \gamma)$

$10(\delta - \gamma) + 10\delta^2$ has the highest utility for $0 < \delta < 1$. However, G_{16} is not a terminal graph when $(\delta - \delta^2) < \gamma < (\delta - \delta^3)$ – but G_{17} is despite having a lower total utility than G_{16}! The condition for $u_T(G_{20}) > u_T(G_{11})$ is $10((\delta - \gamma) + 10\delta^2) > 8(\delta - \gamma) + 12\delta^2$ which reduces to $(\delta - \delta^2) > \gamma$. So when $\gamma > (\delta - \delta^2)$, G_{20} does not dominate G_{11}. As edges are added to graphs, and we move down Table 10, graphs with l edges dominate graphs with $l - 1$ edges if $(\delta - \delta^2) > \gamma$. For γ in this range, the complete graph G_{34} has the highest total utility. However, with $\gamma > (\delta - \delta^2)$ this is no longer the case. The cycle (G_{20}) dominates the other graphs with five edges and graphs with fewer edges when $(\delta - \delta^2) < \gamma$. And the star ($G_{11}$) dominates graphs with fewer edges. The four graphs, in addition to the cycle, not anticipated by JW are G_{12}, G_{17}, G_{19}, and G_{26}. There are four other graphs with six edges that have the same total utility as G_{26} but they are not stable equilibria. Having G_{12} an destination graph for $\delta > \gamma > \delta - \delta^3$ is interesting as it is suboptimal[22] compared to G_{11}. Again, the sequencing of decisions by satisficing actors is critical in generating network structures and suboptimal structures can be reached.

5. Summary and discussion

Given that networks get generated and many networks have distinctive structures, it is reasonable to examine the mechanisms whereby these structures are generated. Jackson and Wolinsky (1996) [JW] suggested a strictly deductive approach involving rational actors and regimes of costs and benefits from maintaining network ties. They were able to establish (Theorem 1) conditions for the emergence of the null graph, star and complete graphs. As such, they provided a challenge to the descriptive mode of pursuing network analysis. Hummon (2000) used an agent-based simulation to explore the dynamics suggested by JW. His results suggested that there were other equilibrium structures, structures not anticipated by JW's formal analysis, and this created an intellectual puzzle.

The primary goal here was to explore this puzzle to see if it is possible to establish the conditions under which equilibrium structures are generated. In so doing, it was necessary to generalize slightly the notion of an equilibrium structure, from one with a maximized total utility, to one that can be reached but that, once there, such network structures persist. Key to this exploration are two ideas: rational actors employ a bounded rationality (Simon, 1976) and the *sequencing* of decisions by such actors is critically important. Theorems 2–4, for graphs with three to five vertices establish the conditions for equilibrium structures. Included are the structures 'predicted' by JW but they are only a subset of possible equilibrium structures, as suggested by Hummon (2000). Theorems 2–4 were established via exhaustive searches, a strategy that is prohibitive for large network structures. However, these searches suggest that 'global theorems' such as those offered by JW may not possible for arbitrarily large networks. The simulations of Hummon (2000) and the detailed examination of the sequence of possible changes suggest that the sequence of decisions made by rational actors is important.[23] Necessarily, these sequences are empirically contingent and early decisions may preclude later decisions for (boundedly) rational actors in ways that are not considered in approaches assuming that actors are completely rational with unlimited informational resources and the time to survey all future options. Even so, the analyses of JW are of great importance in provoking questions about how networks are generated. Given their pioneering work it is no longer possible to simply describe network structures.

Clearly, there are limitations to the current analyses. As noted above, exhaustive searches are prohibitive – although looking at the lattice of networks where $n = 6$ is a tempting next step. While the JW approach, in general terms, is suggestive, some of the simplifications may merit additional attention. First, we assumed that the self-evaluations w_{ii} are identical and this could be relaxed. Second, and more consequentially, the assumptions that costs (γ_{ij}) and benefits (δ_{ij}) are the same $\forall i, j$ seem overly restrictive. If different actors do have

22 The condition for $u_T(G_{12}) > u_T(G_{11})$ is $\delta^3 > \delta^2$ which is impossible for $0 < \delta < 1$.

23 The idea that certain structures may preclude or inhibit specific other structures in evolutionary sequences of structures is evident also in Doreian and Krackhardt's (2001) analyses of signed graphs through time.

different regimes of costs and benefits, this can be included and so avoid the assumption of 'universal' actors – although this will be more easily done within simulations than with analytic derivations. A variation on the idea of differential δ_{ij} (and γ_{ij}) is that such differences are due to structural configurations rather than actor differences. The arguments of Burt (1992) suggest that certain ties may have greater utility if they bridge structural holes. (This has the implication that benefits and costs may be time varying.) In principle, such ideas can be incorporated into the analyses conducted here, albeit at the cost of greater complexity.

No attempt was made herein to examine whether hierarchies are more likely to form that other structures. Of course, the star can be viewed as a hierarchical structure – but neither the cycle nor the complete graph can be viewed in this fashion. Gould (2002) focused explicitly on the conditions for the evolution of hierarchies and this line of thought could be included in the current analysis.

As noted in Section 1, there is an implicit assumption that the framework applies in an unmodified form to networks of all sizes. This seems problematic. Either through preferences (e.g. Zeggelink et al. (1997)) or the constraints of time it is not possible for (most) actors to sustain an indefinitely large number of ties. The assumptions of the above analyses can be relaxed to allow actors to maintain a different number of ties and to constrain the maximum number of network ties maintained by a single actor. One consequence will be that the transitions will have to be considered in a semilattice rather than a lattice. Another is that the analyses will be far more complicated in formal analyses of the sort proposed by JW than in simulation studies of these network dynamics. One complication in this context, is that is may be necessary to consider declining marginal utilities as additional ties are accumulated by an actor.

The device of considering changes in networks through the addition (or deletion) of single links is a simplification that makes the above analyses in terms of a lattice possible. Changes could occur simultaneously rather than sequentially. Changes of two ties at a time could be incorporated into a different lattice, as could a simultaneous inclusion of one and two tie changes. However, the difficulties would be considerable. Another potential drawback is that geodesics are particularly important in the current analyses: shorter paths eclipse longer paths in the basic utility equations. If longer paths between i and j still have utility even after shorter paths between i and j have been established, this can be incorporated into the utility functions.

While there are multiple ways of extending the current analyses, it is unlikely that the two basic insights from them (and the Hummon simulations) will be diminished: different regimes of costs and benefits will lead to the generation of different structures and the sequence by which actors makes decisions, and so help generate structures, are critical. Of course, these assertions can always be made – but both the formal analyses and simulation studies considered here provide disciplined support for the assertions.

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