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Short communication

A note on actor network utilities and network evolution

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Abstract

The corrections provided by Xie and Cui [Xie, F., Cui, W., in press. Cost range and network structures. Social Networks, 30] to one of the theorems provided by Doreian [Doreian, P., 2006. Actor network utilities and network evolution. Social Networks, 28, 137–164] are both necessary and helpful. Accepting these corrections, and linking them more closely to the work of Hummon [Hummon, N.P., 2000. Utility and dynamic social networks. Social Networks, 22, 221–249], leads to some additional suggestions which, together with a restatement of some earlier results, help set the foundations for future work in this area.

Keywords: Networks structure; Network evolution; Network utilities

I greatly appreciate the close attention that Xie and Cui (in press) have given to an earlier paper of mine (Doreian, 2006) and welcome the corrections that they have proposed for my Theorem 4 therein. All of the results in their paper and mine rest on the simplified expression for utilities that was based on the formulation by Jackson and Wolinsky (1996). Given that actors in networks do make choices about creating and dissolving their social ties with others, the natural question arises: what kinds of structures get generated through these choices? Jackson and Wolinsky's formulation provided an elegant way of approaching this question. Notationally, δ is the direct benefit to i from the tie to j, γ is the cost for maintaining the tie (where δ and γ are the same for all actors). Further, $\delta^{t_{ij}}$ is the indirect benefit to i from being linked to j over geodesics of length t_{ij} (when there are no shorter paths linking the two vertices).

$$u_i(G) = w + \sum_{ij\epsilon G}(\delta - \gamma) + \sum_{t_{ij}>1}(\delta^{t_{ij}})$$

Examining whether or not a transition is made from one graph to another, via the addition or deletion of a tie, depends on the comparison of the utilities of actors in the two graphs. In these comparisons, the w terms are subtracted out in the comparison. For notational simplicity, all utility expressions that are written out have the w terms omitted.

The service and reminder (at least to me) provided by Xie and Cui is to look closely at the *deletion* of ties as well as their creation, something present in the Hummon (2000) simulation study and not considered closely enough in my paper. This is made very clear by considering Fig. 1 (which was Fig. 8 in my earlier paper) where the values listed on the left give the number of edges in the graphs on the same line. All of 34 five-vertex edge graphs, G_i, for $1 \le i \le 34$, are shown in Fig. 6 in Doreian (2006) and those mentioned in the narrative here are shown in Xie and Cui (in press). The solid lines represent those transitions that are possible, given the parameter constraints specified for the figure, while the dashed lines represent transitions that cannot be made for these parameter constraints. The solid lines leading to G_{17}, contrary to my earlier claim, do not mean that this graph is a stable equilibrium even though it can be reached over several pathways of transitions. There is also the dashed line from G_{12} to G_{17} representing a transition that could not occur. This implies, as Xie and Cui demonstrate, that G_{17} will move to G_{12} through a line removal. The same argument applies to my earlier Fig. 9 where G_{19} cannot be stable for $\delta - \delta^2 < \gamma < \delta$ because the impossible transition from G_{12} to G_{19} implies a movement from G_{19} to G_{12}, if G_{19} is reached over other pathways, for these parameter constraints through line removal. Both Figs. 8 and 9 in Doreian (2006) remain correct and the figures remain useful 'inference devices' – for creating the components of theorems linking equilibrium structures to ranges of γ – only if *line deletion is considered as well as line formation*. Xie and Cui's result, expanding the parameter range for γ where the null graph can be

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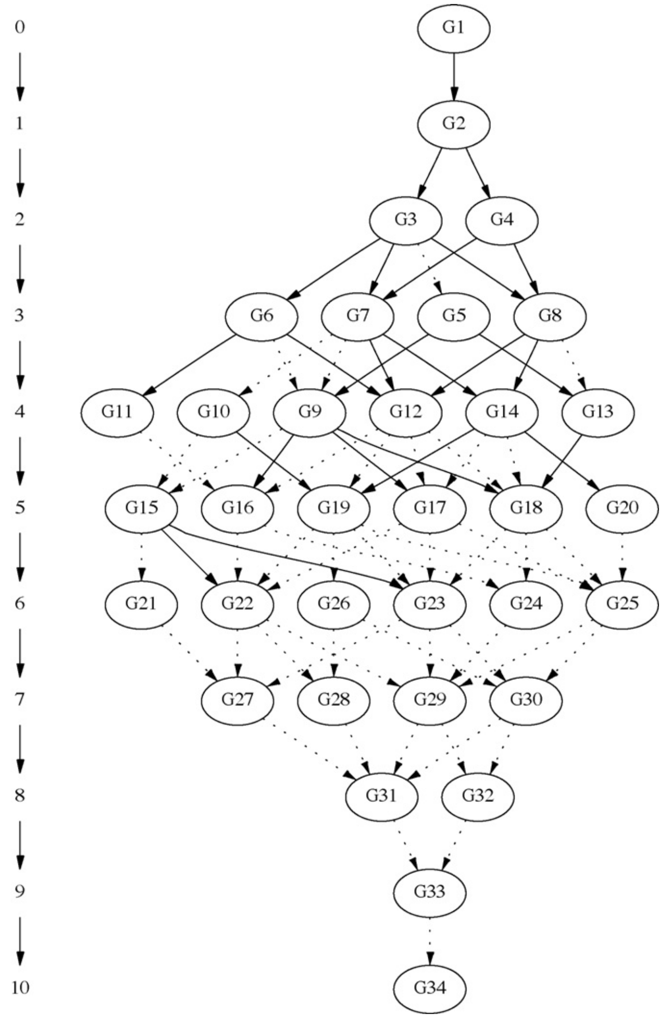

Fig. 1. Transitions between five-vertex graphs for $\delta - \delta^2 < \gamma < \delta - \delta^3$.

an equilibrium structure, reenforces the need to consider edge deletion.

Given the corrections to Theorem 4 provided by Xie and Cui (in press), it is useful to reconsider the work of Hummon (2000). He identified eight graphs in a 'structural typology' of graphs: null, near-null, star, near-star, shared (cycle), near-shared, near-complete and complete (Hummon, 2000, p. 234). These graphs constitute the set of equilibrium structures from his simulations. The null, star, cycle and complete graphs all appear in the corrected Theorem 4. So, too, does the near-star (as G_{12}). Hummon defined a measure of fit for a graph, relative to any member of his structural typology, by using the degree sequence of a graph. Letting a vector $d = (d_1, \ldots, d_i, \ldots d_n)$ and $o = (o_1, \ldots o_i, \ldots o_n)$ denote the degree sequences of an ideal and observed graph respectively, the measure of fit is $\sum_i (o_i - d_i)^2/n$. Using this measure of fit, there are six graphs equally close to G_{20}. One of them is G_{26} (which is an equilibrium structure identified by Xie and Cui and by Doreian) and can be interpreted as the near-cycle. (The other five graphs that are equally close to the cycle structure are not equilibria.)

The corrected version of Theorem 4 as given by Xie and Cui (in press), with the equilibrium graphs named, is:

Theorem 1. *For graphs on five vertices, the equilibrium structures are*:

1. *For $\gamma > \delta + \delta^2 - \delta^3 - \delta^4$, only the null graph ($G_1$) results*;
2. *For $\delta + \delta^2 - \delta^3 - \delta^4 > \gamma > \delta$, the null graph ($G_1$) and the cycle ($G_{20}$) result*;
3. *For $\delta > \gamma > \delta - \delta^3$, the star ($G_{11}$), the cycle ($G_{20}$) and the near-star ($G_{12}$) result.*
4. *For $\delta - \delta^3 > \gamma > \delta - \delta^2$, the star ($G_{11}$), the cycle ($G_{20}$) and the near-cycle ($G_{26}$) result*; *and*
5. *For $\gamma < \delta - \delta^2$ only the complete graph (G_{34}) results.*

It would appear the neither the near-null nor the near-complete structures identified by Hummon (2000) are among the equilibrium structures for edge graphs on five vertices. The argument by Xie and Cui (in press) regarding the conditions under which a cycle (G_{20}) could become a null graph (G_1) rules out the near-null graph (item 2 of the corrected theorem) and a similar argument applies to the near-complete graph. At this point, it is important to note that Hummon (2000) considered a variety of decision rules (four) within a basic division between mutual tie formation where both actors have to benefit from the formation of a tie and unilateral tie imposition where one actor can impose a tie on another actor. All of the cases considered here are within the rubric of mutual tie formation and mutual tie severance which suggests that the near-null and near-complete structures can emerge only when ties can be imposed unilaterally.

Xie and Cui (in press) observe that the cycle exists as an equilibrium structure over a broad range for the constraints on γ. This is an important insight, one anticipated by Hummon who wrote "mutual rules tend to promote shared [cycle] and near-shared [near-cycle] structures" (Hummon, 2000, p. 247) as part of his summary of his simulation results. Theorem 4, as corrected by Xie and Cui, makes this statement more precise. It also reinforces the notion that *the decision rules are critical* regarding the equilibrium structures that can be generated via choices by actors in a network.

Xie and Cui note that the star has the next longest range for γ in which it can exist as an equilibrium structure. This points to a problem with the utility formulation considered here that lurks beneath the surface of utility analyses such as these. Consider the star structure, G_{11} for graphs with five vertices, and the utilities for this structure. Denoting the central vertex as c and each of the peripheral vertices as p, with n the number of vertices, the utilities are $u_c = (n-1)(\delta - \gamma)$ and $u_p = (\delta - \gamma) + (n-2)\delta^2$. Hummon (2002, p. 236-6) notes the $u_c \leq u_p$ in general (they are the same only if $\gamma = \delta - \delta^2$). If c is a parent and each p is a child of c, this result might seem reasonable. But if the star is thought as a hierarchy where the top actor, c, derives the most benefit while each p benefits the least, this result seems unreasonable. Hummon shows that in order to make the values of u_c and u_p the same, a transfer payment of $((u_p - u_c)/n)$ is required. This implies a transfer payment of $((n-2)/n)(\delta^2 - (\delta - \gamma))$ from each p to c. Looking at this slightly differently, the condition under which $(u_c) > (u_p)$ is $\delta - \delta^2 > \gamma$. However, this is the condition for a complete graph forming (item 5 of the corrected theorem). This implies a star, where *c* derives the most benefit, will not form absent the presence of side payments. This merits further attention.

Examining utilities within this framework can be linked to the notion of equivalence in a network, especially automorphic equivalence and structural equivalence as a special case. Two vertices in the same orbit of a graph are automorphically equivalent (Everett and Borgatti, 1988). Actors choose to form or sever ties in a network depending on the gains or losses of either course of action. Given that a network is in some state, the formation of a tie or the deletion of a tie changes the decision context for all actors. This suggests that structurally equivalent actors and, more generally, automorphically equivalent actors are competitors for making a choice about forming a tie if there are benefits to be had from the tie's formation. Burt (1982) introduced the concept of structural holes explicitly with regard to strategic action to derive benefits from changing locations in a network to capitalize on filling structural holes. These aspects of structure can be linked explicitly to utility analyses.

In his simulations, Hummon (2000) considered, at a choice point for an actor, all of the choices this actor could make, including the option to neither form nor delete a tie. However, the formation or deletion of a tie between a pair of actors often have implications for third parties who might gain or lose utility from the actions of others regarding ties. Part of the strategy, for an actor, thinking about forming a tie – or not – is a consideration of the potential actions of others. This can be influenced by third parties. Following this line of thought, it is possible that another kind of side payment can be envisioned where an actor, say *k*, can offer a side payment to *i* and *j* or one of them) to form a tie between *i* and *j* from which *k* can benefit as a third party.

Hummon (2000, p. 247) also notes that "group size matters" given the differences he saw in his simulation results for groups of sizes 3, 4, 5 and 10. Clearly, the differences between Theorems 2 and 3 of Doreian (2006) and the corrected Theorem 4 (Theorem 1 above) show differences in equilibrium structures as the group size goes from 3 to 5. Hummon goes on to claim "the differences do not suggest specific trends that could be extrapolated to larger groups". While it is not possible to simulate processes for groups of all sizes, such extrapolations seem necessary. It is clear also that the detailed comparisons of Doreian (2006) and of Xie and Cui (in press) are prohibitive for large group sizes. Yet the need for specifications for the structural outcomes to larger group sizes remains. Some judicious combination of the two approaches seems necessary.

The various theorems that have resulted from adopting the approach of Jackson and Wolinsky (1996) provide clear results—but only for the 'stylized structures' considered here. It is one thing to write about benefits (δ) and costs (γ) but it is another matter to measure them across the actors of a group (or actors who could form a group). It is likely that both δ and γ vary across individuals and through time for individuals and this needs to be considered in future work seeking empirical application of these ideas. Even so, the kinds of corrections provided by Xie and Cui help set the foundations properly for that work. Much remains to be done.

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